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Fuzzy point hyper BCK-algebras

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Abstract

By using the concept of fuzzy points, we generalize the notion of hyper BCK-algebra and the notions of fuzzy point hyper BCK-(sub) algebras, fuzzy point (weak, strong) hyper BCK-ideals, quasi hyper BCK-(sub) algebras and quasi (weak, strong) hyper BCK-ideals are introduced. The relationship between these notions are stated and proved. Finally, we study the condition QH on quasi hyper BCK-algebras.

Keywords: Hyper BCK-algebra, (quasi, fuzzy point) hyper BCK-algebra, (quasi, fuzzy point) hyper BCK-ideal.

Mathematics Subject Classification: 06F35, 03G25.

Introduction

The hyperalgebraic structure theory was introduced by Marty (1934). Imai and Iseki (1966) introduced the notion of a BCK-algebra. Recently, Jun *et al*. (2000) applied the hyperstructure to BCK-algebras and introduced the concept of the hyper BCK-algebra which is a generalization of the BCK-algebra.

It is well-known that the category of bounded commutative BCK-algebras is equivalent to the category of MV-algebras. In particular, any bounded commutative BCK-algebra is an MV-algebra and *vice versa*. On the other hand, an MV-algebra is an algebraic structure of the Lukasiewicz many-valued logic. Hence any bounded commutative BCK-algebra is somehow related to a many-valued logic.

Since the concept of the hyper BCK-algebra is a generalization of the notion of the BCK-algebra, it is natural to search for logic whose algebraic structure is a hyper BCK-algebra. To this end, we first need a deeper understanding of hyper BCK-algebras. This structure was studied in (Borzooei et. al 2002, Borzooei et. al 2003, Jun et al 2001and Jun et al 2000).

Here we use the notion of fuzzy point to establish the notions of fuzzy point hyper BCK-(sub) algebras, fuzzy point (weak, strong) hyper BCK-ideals, quasi hyper BCK-(sub) algebras and quasi (weak, strong) hyper BCK-ideals, then we obtain some related results which have been mentioned in the abstract.

Preliminaries

Definition 2.1. (Jun, Zahedi et al 2000) Let H be a nonempty set and o be a hyperoperation on H, that is o is a function from $H \times H$ to $P^*(H) = P(H) \setminus \{\emptyset\}$. Then H is called a hyper BCK-algebra if it contains a constant 0 and satisfies the following axioms:

$(HK1)$ $(x \, o \, z) \, o \, (y \, o \, z) << x \, o \, y,$

$(HK2)$ $(x \, o \, y) \, o \, z = (x \, o \, z) \, o \, y,$

$(HK3)$ $x \, o \, H << \{x\},$

$(HK4)$ $x << y$ *and* $y << x$ *imply* $x = y,$

For all $x, y, z \in H$ where $x << y$ is defined by $0 \in x \, o \, y$.

For every $A, B \subseteq H$, $A << B$ is defined by $\forall a \in A, \exists b \in B$ such that $a << b$, and we have $A o B = \bigcup_{a \in A, b \in B} a o b$.

Proposition 2.2. (Jun, Zahedi et al 2000) *In any hyper BCK*-algebra H, the following statements hold:

$(i)\, 0 o 0 = \{0\},$
$(ii)\, 0 << x,$
$(iii)\, x << x,$
$(iv)\, 0 o x = \{0\},$
$(v)\, x \, o \, y << x,$
$(vi)\, x \in x \, o \, 0,$
$(vii)\, A << 0$ *implies* $A = \{0\},$
$(viii)\, x o 0 << \{y\}$ *implies* $x << y$,
$(ix)\, y << z$ *implies* $x \, o \, z << x \, o \, y.,$
for all $x, y, z \in H$ *and* $A \subseteq H.$

Proposition 2.3. In any hyper BCK-algebra H, $x o 0 = \{x\}$ for all $x \in H$.

Proof. We have $x \in x\,o\,0$. Now, let $t \in x\,o\,0$. Since $x\,o\,0 << \{x\}$ we have $t << x$, so $0 \in t\,ot \subseteq (x\,o\,0)ot = (x\,o\,t)o\,0$, Then there exists a $h \in x\,ot$ such that $0 \in h\,o\,0$ thus $h << 0$, hence $h = 0$, then $x << t$. We conclude that $x = t$. Therefore $x\,o\,0 = \{x\}$.

Definition 2.4. (Jun *et al*., 2000) Let $(H, o, 0)$ be a hyper BCK-algebra and S be a subset of H. If S is a hyper BCK-algebra with respect to the hyper operation "o" on H, we say that S is a hyper subalgebra of H.

Definition 2.5. (Jun *et al*., 2000) Let I be a nonempty subset of a hyper BCK-algebra $(H, o, 0)$ and $0 \in I$. Then I is called a (weak, strong) hyper BCK-ideal of H if $((xoy) \subseteq I, (x\,o\,y) \cap I \neq 0)$ $(x\,o\,y) <<$ I and $y \in I$ imply that $x \in I$, for all x, y $\in H$.

Definition 2.6. (Jun *et al*., 2000) A hyper BCK-ideal I of H is said to be reflexive if $x\,o\,x \subseteq I$, for all $x \in H$.

Lemma 2.7. (Jun *et al*., 2000) Let A, B and I be subsets of H.

(i) If $A \subseteq B << C$, then $A << C$.

(ii) If $A\,o\,x << I$ for $x \in H$, then $a\,o\,x << I$ for all $a \in A$.

(iii) If I is a hyper BCK-ideal of H and if $Aox << I$ for $x \in I$, then $A << I$

(iv) Let I be a reflexive hyper BCK-ideal of H and let A be a subset of H. If $A << I$, then $A \subseteq I$.

Definition 2.8. (Jun *et al*., 2000) Let H be a hyper BCK-algebra. Define the set $\nabla(a,b) := \{x \in H | 0 \in (xoa)ob\}$. If for any $a, b \in H$, the set $\nabla(a,b)$, has the greatest element (with respect to <<) then we say that H satisfies the hyper condition.

We now review some fuzzy logic concept (see (Borzooei and Zahedi 2002 and Zadeh 1965)).

A fuzzy set μ in a set X of the form

$$\mu(y) := \begin{cases} t & if y = x, \\ 0 & otherwise, \end{cases}$$

Where $t \in (0.1]$, is called a fuzzy point with support x and value t and is denoted by x_t

Definiton 2.9. (Jun et al 2001) Let μ be a fuzzy set of H. Then

(i) μ is called a fuzzy hyper BCK-ideal of H if

(a) $x << y$ implies that $\mu(x) \geq \mu(y)$,

(b) $\mu(x) \geq \min\{\inf_{a \in xoy} \mu(a), \mu(y)\}$,

(ii) μ is called a fuzzy weak hyper BCK-ideal of H if $\mu(0) \geq \mu(x) \geq \min\left\{\inf_{a \in xoy} \mu(a), \mu(y)\right\}$,

(iii) μ is called a fuzzy strong hyper BCK-ideal of H if

$$\inf_{a \in xoy} \mu(b) \geq \mu(x) \geq \min\left\{\sup_{a \in xoy} \mu(a), \mu(y)\right\},$$

for all $x, y, z \in H$.

Definition 2.10. (Borzooei and Zahedi 2002) Fuzzy set μ satisfies the additive condition, if $x << y$ implies that $\mu(x) \geq \mu(y)$..

Definition 2.11. (Jun et al 2001) Let μ be a fuzzy set in H. Then μ is called a fuzzy hyper BCK-subalgebra of H if

$$\inf_{s \in (xoy)} \mu(s) \geq \min\{\mu(x).\mu(y)\}.$$

Quasi Hyper BCK-algebra

From now on $(H, o, 0)$ is a hyper BCK-algebra, unless otherwise is stated.

Let $F(H) = \{x_a | x \in H, a \in (0,1]\}$. We define the hyperoperation Θ on F (H) by:

$$\Theta : F(H) \times F(H) \rightarrow F(H)$$

$$(x_\alpha, y_\beta) \mapsto \{z_{\min\{\alpha,\chi\}} | z \in xoy\}$$

Where $(xoy)_\gamma = \{z_\gamma \mid z \in xoy\}$,

We define the relation $\prec$ on F(H) by:

$x_\alpha \prec y_\beta$ if and only if $0_{\min\{\alpha,\beta\}} \in (xoy)_{\min\{\alpha,\beta\}}$.

Let A, B be any subsets of F(H). We define $A \prec B$ if and only if for all $a \in A$ there exists $b \in B$ such that $a \prec b$.

Lemma 3.1. Let F(H) be as above with operation Θ and order $\prec$. Then we have:

(i) $(x_\alpha \Theta z_\beta)\Theta(y_\gamma \Theta z_\beta) \prec x_\alpha \Theta y_\gamma$,

(ii) $(x_\alpha \Theta y_\beta)\Theta z_\gamma = (x_\alpha \Theta z_\gamma)\Theta y_\beta$,

(iii) $x_\alpha \Theta F(H) \prec \{x_\alpha\}$.

for all $x_\alpha, y_\gamma, z_\beta \in F(H)$.

Proof. (i) Consider

$$(x_\alpha \Theta z_\beta)\Theta(y_\gamma \Theta z_\beta) = (xoz)_{\min\{\alpha,\beta\}} \Theta (yoz)_{\min\{\gamma,\beta\}}$$

$$= \bigcup_{t \in xoz, h \in yoz} (toh)_{\min\{\alpha,\beta,\gamma\}}$$

$$= ((xoz)o(yoz))_{\min\{\alpha,\beta,\gamma\}.}$$

For all $s_{\min\{\alpha,\beta,\gamma\}} \in ((xoz)o(yoz))_{\min\{\alpha,\beta,\gamma\}}$ we get that $s \in ((xoz)o(yoz))$. since H is a hyper BCK-algebra, then

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there exists a $k \in xoy$ such that s<<k therefore $0 \in sok$ and $k_{\min\{\alpha,\gamma\}} \in (xoy)_{\min\{\alpha,\gamma\}}$. In other hand $s_{\min\{\alpha,\beta,\gamma\}} \Theta k_{\min\{\alpha,\gamma\}} = (sok)_{\min\{\alpha,\beta,\gamma\}}$, then we get that $0_{\min\{\alpha,\beta,\gamma\}} \in (sok)_{\min\{\alpha,\beta,\gamma\}} = s_{\min\{\alpha,\beta,\gamma\}} \Theta k_{\min\{\alpha,\gamma\}}$. Hence $s_{\min\{\alpha,\beta,\gamma\}} \prec k_{\min\{\alpha,\gamma\}}$ which imply that $(x_\alpha \Theta z_\beta)\Theta(y_\gamma \Theta z_\beta) \prec x_\alpha \Theta y_\gamma$.

(ii) Since H is a hyper BCK-algebra, then we have

$$(x_\alpha \Theta y_\beta)\Theta z_\gamma = (xoy)_{\min\{\alpha,\beta\}} \Theta z_\gamma = \bigcup_{t \in toy} (toz)_{\min\{\alpha,\beta,\gamma\}} = \bigcup_{h \in (xoy)oz} h_{\min\{\alpha,\beta,\gamma\}} = \bigcup_{h \in (xoz)oy} h_{\min\{\alpha,\beta,\gamma\}} = (x_\alpha \Theta z_\gamma)\Theta y_\beta.$$

(iii) We have $x_\alpha \Theta F(H) = \bigcup_{z_\gamma \in F(H)} x_\alpha \Theta z_\gamma = \bigcup_{z \in H} (xoz)_{\min\{\alpha,\gamma\}}$. Consider $t_\beta \in x_\alpha \Theta F(H)$. Then there exists $z_\gamma \in F(H)$ such that $t_\beta \in x_\alpha \Theta z_\gamma = (xoz)_{\min\{\alpha,\gamma\}}$. Hence $t \in (xoz)$ and $\beta = \min\{\alpha,\gamma\}$. On the other hand we have $xoz << \{x\}$, thus for any $t \in xoz$, we have t<<x, then $0 \in tox$. So $0_{\min\{\alpha,\gamma\}} \in (tox)_{\min\{\alpha,\gamma\}} = t_\beta \Theta x_\alpha$. Therefore $t_\beta \prec x_\alpha$ and we get that $x_\alpha \Theta F(H) \prec \{x_\alpha\}$.

Let $x_\alpha \prec y_\beta$ and $y_\beta \prec x_\alpha$. Then we get that $0_{\min\{\alpha,\beta\}} \in (xoy)_{\min\{\alpha,\beta\}}$ and $0_{\min\{\alpha,\beta\}} \in (yox)_{\min\{\alpha,\beta\}}$, so $0 \in xoy$ and $0 \in yox$, then x=y, by (HK4). But if $\alpha = \beta$, then we can conclude that $x_\alpha = y_\beta$, on the otherwise we haven't $x_\alpha = y_\beta$.

For example in a hyper BCK-algebra X, we have $0_{0.1} \in (bob)_{\min\{0.6,0.1\}} = b_{0.1}\Theta b_{0.6}$ and $0_{0.1} \in (bob)_{\min\{0.6,0.1\}} = b_{0.6}\Theta b_{0.1}$, but $b_{0.1} \neq b_{0.6}$.

Hence F(H) is not a hyper BCK-algebra under the operation ⊙, since (HK4) does not hold. We call it a quasi hyper BCK-algebra.

Proposition 3.2. In the quasi hyper BCK-algebra F(H) the followings hold for all $x, y \in H, \alpha, \beta, \gamma \in (0,1]$ and $A \subseteq F(H)$:

(i) $0_\alpha \Theta 0_\beta = \{0_{\min\{\alpha,\beta\}}\}$,

(ii) $0_\alpha \prec x_\beta$,

(iii) $x_\alpha \prec x_\beta$,

(iv) $A \prec A$,

(v) $0_\alpha \Theta x_\beta = \{0_{\min\{\alpha,\beta\}}\}$

(vi) $0_\alpha \Theta A = \{0_{\min\{\alpha,\beta\}} \mid \beta \in (0,1], \text{ for all } x_\beta \in A\}$

(vii) $A \prec \{0_\alpha\}$ implies that $A = \{0_\beta \mid \beta \in (0,1] \text{ and } x_\beta \in A\}$

$(viii)$ $x_\alpha \Theta 0_\beta = \{x_{\min\{\alpha,\beta\}}\}$,

(ix) $x_\alpha \Theta 0_\beta \prec \{y_\gamma\}$ implies that $x_\alpha \prec y_\gamma$,

(x) $x_\alpha \prec y_\beta$ if and only if $x_\beta \prec y_\alpha$,

(xi) $y_\beta \prec z_\gamma$ implies $x_\alpha \Theta z_\gamma \prec x_\alpha \Theta y_\beta$,

(xii) $A\Theta 0_\beta \prec \{0_\gamma\}$ implies $A = \{0_\alpha \mid \alpha \in (0,1] \text{ and } x_\alpha \in A\}$,

$(xiii)$ $x_\alpha \Theta 0_\beta \prec \{x_\gamma\}$.

Proof. The proof is straightforward and follows from Proposition 2.2.

Definition 3.3. Let T be a subset of the quasi hyper BCK-algebra F(H). Then T is called.

(i) a quasi hyper BCK-subalgebra of F(H), if $x_\alpha \Theta y_\beta \subseteq T$, for all $x_\alpha, y_\beta \in T$,

(ii) a quasi (weak, strong) hyper BCK-ideal of F(H), if

(a) $0_\alpha \in T$, for all $\alpha \in (0,1]$

(b) $((x_\alpha \Theta y_\beta) \subseteq T, (x_\alpha \Theta y_\beta) \cap T \neq \emptyset)(x_\alpha \Theta y_\beta) \prec T$ and $y_\beta \in T$ imply that $x_\alpha \in T$, for all $x_\alpha, y_\beta \in F(H)$.

For any $\alpha \in (0,1]$, we define $F_\alpha(H) := \{x_\alpha \mid x \in H\}$. Obviously, $(F_\alpha(H), \Theta, 0_\alpha)$ is a hyper BCK-algebra and can be identified it with H. We call it a fuzzy point hyper BCK-algebra.

Definition 3.4. Let $\alpha \in (0,1]$ and T_α be a subset of fuzzy point hyper BCK-algebra $F_\alpha(H)$. Then T_α is called

(i) a fuzzy point hyper BCK-subalgebra, if $x_\alpha \Theta y_\alpha \subseteq T_\alpha$, for all $x_\alpha, y_\alpha \in T_\alpha$,

(ii) a fuzzy point (weak, strong) hyper BCK-ideal of $F_\alpha(H)$, if

(a) $0_\alpha \in T_\alpha$,

(b) $((x_\alpha \Theta y_\alpha) \subseteq T_\alpha, (x_\alpha \Theta y_\alpha) \cap T_\alpha \neq \emptyset)(x_\alpha \Theta y_\alpha) \prec T_\alpha$ and $y_\alpha \in T_\alpha$ imply that $x_\alpha \in T_\alpha$, for all $x_\alpha, y_\alpha \in F_a(H)$.

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Example 3.5. Let H={0, a, b}. Then the following table shows a hyper BCK-algebra structure on H.

o	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

Let $\alpha \in (0,1]$. Then $\{0_\alpha, a_\alpha\}$ and $\{0_\alpha, b_\alpha\}$ are fuzzy point hyper BCK-subalgebra and fuzzy point (weak, strong) hyper BCK-ideal of $F_\alpha(H)$.

Theorem 3.6. Let $T \subseteq H, \alpha \in (0,1]$ and $T_\alpha \subseteq F_\alpha(H)$. Then

(i) T is a subalgebra of H if and only if T_α is a fuzzy point subalgebra of $F_\alpha(H)$.

(ii) T is a (weak, strong) hyper BCK-ideal of H if and only if T_α is a fuzzy point (weak, strong) hyper BCK-ideal of $F_\alpha(H)$,

(iii) T is a (weak, strong) hyper BCK-ideal of H if and only if T_α is quasi (weak, strong) hyper BCK-ideal of F(H).

Proof. (i) Let $x_\alpha, y_\alpha \in T_\alpha$. Then $x, y \in T$, and since T is a sublagebra of H we have $xoy \subseteq T$. So $(xoy)_\alpha \subseteq T_\alpha$, hence $(x_\alpha \Theta y_\alpha) \subseteq T_\alpha$.

Conversely, let $x, y \in T$. Then $x_\alpha, y_\alpha \in T_\alpha$, and since T_α is a fuzzy point subalgebra of $F_\alpha(H)$, we have $(x_\alpha \Theta y_\alpha) \subseteq T_\alpha$. Thus $(xoy)_\alpha \subseteq T_\alpha$ and so $xoy \subseteq T$.

The proof of (ii) and (iii) are similar to the above argument.

Proposition 3.7. Let $\alpha \in (0,1]$. Then $F_\alpha(H)$ is a quasi hyper BCK – subalgebra of F(H).

It is clear that $F_\alpha(H)$ is not a (weak, strong) hyper BCK-ideal. If $\beta > \alpha$, then $x_\beta \Theta y_\alpha = (xoy)_\alpha \prec F_\alpha(H)$ and $y_\alpha \in F_\alpha(H)$, but $x_\beta \notin F_\alpha(H)$.

Theorem 3.8. Let F(H) be a quasi hyper BCK-algebra. Then

$$S(F(H)) := \{x_\alpha \in F(H) \mid x_\alpha \Theta x_\alpha = \{0_\alpha\}\}$$

is a quasi hyper BCK-subalgebra of F(H).

Proof. Since $0_\alpha \Theta 0_\alpha = \{0_\alpha\}$, then we get that S(F(H)) is nonempty. Now, let $x_\alpha, y_\beta \in S(F(H))$. By Lemma 3.1, we have $(x_\alpha \Theta y_\beta)\Theta(x_\alpha \Theta y_\beta) \prec x_\alpha \Theta x_\alpha = \{0_\alpha\}$ and so $((xoy)o(xoy))_{\min\{\alpha,\beta\}} \prec \{0_\alpha\}$. Thus by Proposition 3.2(vii) we have $((xoy)o(xoy))_{\min\{\alpha,\beta\}} = \{0_{\min\{\alpha,\beta\}}\}$.

Now, let $a_{\min\{\alpha,\beta\}} \in x_\alpha \Theta y_\beta = (xoy)_{\min\{\alpha,\beta\}}$. Then $a \in xoy$ and so $ao\,a \subseteq ((xoy)o(xoy))$. Thus.

$a_{\min\{\alpha,\beta\}} \Theta a_{\min\{\alpha,\beta\}} = (aoa)_{\min\{\alpha,\beta\}} \subseteq ((xoy)o(xoy))_{\min\{\alpha,\beta\}} = \{0_{\min\{\alpha,\beta\}}\}$.

Hence $a_{\min\{\alpha,\beta\}} \in S(F(H))$ and $x_\alpha \Theta y_\beta \subseteq S(F(H))$. Therefore S(F(H)) is a quasi hyper BCK-subalgebra of F(H).

Now we show that S(F(H)) satisfies in some condition of BCK-algebra.

Theorem 3.9. The set $x_\alpha \Theta y_\beta$ is singleton, for all $x_\alpha, y_\beta \in (S(F(H)), \Theta)$.

Proof. Let $x_\alpha, y_\beta \in S(F(H))$ and $a_{\min\{\alpha,\beta\}}, b_{\min\{\alpha,\beta\}} \in x_\alpha \Theta y_\beta = (xoy)_{\min\{\alpha,\beta\}}$. By the proof of Theorem 3.8, we have $(aob)_{\min\{\alpha,\beta\}} \subseteq ((xoy)o(xoy))_{\min_{\{\alpha,\beta\}}} = \{0_{\min\{\alpha,\beta\}}\}$ and so $(aob)_{\min\{\alpha,\beta\}} = \{0_{\min\{\alpha,\beta\}}\}$. Thus $0 \in aob$ and similarly we get that $0 \in boa$. Hence $a_{\min\{\alpha,\beta\}} = b_{\min\{\alpha,\beta\}}$.

We call S(F(H)) the quasi BCK-part of quasi hyper BCK-algebra F(H).

Proposition 3.10. If $((x_\alpha \Theta y_\beta)\Theta(x_\alpha \Theta z_\gamma)) \prec z_\gamma \Theta y_\beta$, for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in (01,]$ then $x_\alpha \Theta x_\beta = \{0_{\min\{\alpha,\beta\}}\}$ and so $F(H) = S(F(H))$.

Proof. Consider

$$\begin{aligned} x_\alpha \Theta x_\beta = (xox)_{\min\{\alpha,\beta\}} &= x_{\min\{\alpha,\beta\}} \Theta x_{\min\{\alpha,\beta\}} \\ &= (x_{\min\{\alpha,\beta\}} \Theta 0_{\min\{\alpha,\beta\}})\Theta(x_{\min\{\alpha,\beta\}} \Theta 0_{\min\{\alpha,\beta\}}) \\ &<< 0_{\min\{\alpha,\beta\}} \Theta 0_{\min\{\alpha,\beta\}} \\ &= 0_{\min\{\alpha,\beta\}} \end{aligned}$$

By Proposition 3.2 (vii) we get that $x_\alpha \Theta x_\beta = \{0_{\min\{\alpha,\beta\}}$

Given a fuzzy set μ in H and $\alpha \in (0,1]$, We define:

$F_\alpha(\mu) := \{x_\alpha \mid \mu(x) \geq \alpha\}$,

and

$$F(\mu) := \bigcup_{\alpha \in (0,1]} F_\alpha(\mu).$$

The following example shows that $F_\alpha(\mu)$ may not be a fuzzy point hyper BCK-subalgebra of $F_\alpha(H)$, for some $\alpha \in (0,1]$.

Example 3.11. In Example 3.5, define fuzzy set μ on H by:

$$\mu(x):\begin{cases}\frac{1}{2} & if\ x=a,\\ 0 & otherwise\end{cases}$$

Then $F_{1/4}(\mu)=a_{1/4}$ and $a_{1/4}\Theta a_{1/4}=0_{1/4}$. We can see that $0_{1/4}\notin F_{1/4}(\mu)$. So $F_{1/4}(\mu)$ is not a fuzzy point hyper BCK - subalgebra of $F_{1/4}(H)$.

Theorem 3.12. Let μ be a fuzzy set in H. Then μ is a fuzzy hyper BCK-subalgebra of H if and only if $F_\alpha(\mu)$ is a fuzzy point hyper BCK-subalgebra of $F_\alpha(H)\ for\ all\ \alpha\in(0,1]$.

Theorem 3.13. Let μ be a fuzzy set in H. Then μ is a fuzzy weak hyper BCK-ideal of H if and only if the nonempty set $F_\alpha(\mu)$ is a fuzzy point weak hyper BCK-ideal of $F_\alpha(\mu)$, for all $\alpha\in(0,1]$.

Proof. Let $F_\alpha(\mu)\neq\emptyset$. Then there exists $x_\alpha\in F_\alpha(\mu)$. Thus $\mu(0)\geq\mu(x)\geq\alpha$ and so $0_\alpha\in F_\alpha(\mu)$. Now, let $x_\alpha\Theta y_\alpha\subseteq F_\alpha(\mu)$ and $y_\alpha\in F_\alpha(\mu)$. Then $\mu(t)\geq\alpha$, for all $t\in xoy$ and $\mu(y)\geq\alpha$. Therefore $\inf\limits_{t\in xoy}\mu(t)\geq\alpha$. Hence by hypothesis we have

$$\mu(x)\geq\min\{\inf_{t\in(xoy)}\mu(t),\mu(y)\}\geq\alpha.$$

Thus $x_\alpha\in F_\alpha(\mu), so\, F_\alpha(\mu)$ is a fuzzy point weak hyper BCK-ideal of $F_\alpha(H)$.

Conversely, let $x\in H$ and $\mu(x)=\alpha$. Then by hypothesis we get that $0_\alpha\in F_\alpha(H)$. So $\mu(0)\geq\alpha=\mu(x)$. Now we must show that $\mu(x)\geq\min\{\inf\limits_{t\in(xoy)}\mu(t),\mu(y)\}$. If $\mu(y)\,or\,\mu(t)=0$ for some $t\in xoy$, then the proof is finished. Suppose that $\mu(y)\,and\,\mu(t)\neq0$ for all $t\in xoy$. Let $\alpha=\min\{\mu(y),\inf\limits_{t\in(xoy)}\mu(t)\}$. Thus $\mu(y)\geq\alpha$ and $\mu(t)\geq\alpha\ for\ all\ t\in xoy$, which imply that $y_\alpha\in F_\alpha(H)$ and $x_\alpha\Theta y_\alpha=(xoy)_\alpha\subseteq F_\alpha(\mu)$. Since $F_\alpha(\mu)$ is a weak hyper BCK-ideal of $F_\alpha(H)$, thus $x_\alpha\in F_\alpha(\mu)$. Therefore $\mu(x)\geq\alpha=\min\{\inf\limits_{t\in(xoy)}\mu(t),\mu(y)\}$.

Theorem 3.14. Let μ be a fuzzy set of H. Then:

(i) if μ is a fuzzy (strong) hyper BCK-ideal of H, then the nonempty set $F_\alpha(\mu)$ is a fuzzy point (strong) hyper BCK-ideal of $F_\alpha(H)$, for all $\alpha\in(0,1]$.

(ii) if μ satisfies the additive condition and the nonempty set $F_\alpha(\mu)$ is a fuzzy point (strong) hyper BCK-ideal of $F_\alpha(H)$, for all $\alpha\in(0,1]$, then μ is a fuzzy (strong) hyper BCK-ideal of H.

Proof. (i) $F_a(\mu)\neq\emptyset$ implies that $0_\alpha\in F_\alpha(\mu)$. Now let $(x_\alpha\Theta y_\alpha)\bigcap F_\alpha(\mu)\neq\emptyset$ and $y_\alpha\in F_\alpha(\mu)$. Then $\mu(y)\geq\alpha$ and there exists $t\in xoy$ such $\mu(t)\geq\alpha$. So $\mu(x)\geq\min\{\sup\limits_{t\in(xoy)}\mu(t),\mu(y)\}\geq\alpha$

Hence $x_\alpha\in F_\alpha(\mu)$. Therefore $F_\alpha(\mu)$ is a fuzzy point strong hyper BCK-ideal of $F_\alpha(H)$.

(ii) Since $xox.<<x$, then by hypothesis we have $\mu(t)\geq\mu(x)$ for all $t\in xox$. Hence $\inf\limits_{b\in xox}\mu(b)\geq\mu(x)$.

Consider min $\left\{\sup\limits_{t\in(xoy)}\mu(t),\mu(y)\right\}=\alpha$. Then for all $\beta,\alpha>\beta$, $\mu(y)>\beta$ and there exists $t_\beta\in xoy$ such that $\mu(t_\beta)>\beta$. So $(x_\beta\Theta y_\beta)\bigcap F_\beta(\mu)\neq\emptyset$ and $y_\beta\in F_\beta(\mu)$ imply that $x_\beta\in F_\beta(\mu)\ for\ all\ \alpha>\beta$. Thus $\mu(x)\geq\beta,\ for\ all\ \beta,\alpha>\beta$. Thus $\mu(x)\geq\alpha=\min\left\{\sup\limits_{t\in(xoy)}\mu(t),\mu(y)\right\}$. Therefore μ is a fuzzy strong hyper BCK-ideal of H.

Proposition 3.15. Let μ be a fuzzy set in H. Then

(i) If $x\in S$, then $x_\alpha\in S(F(H)),\ for\ all\ \alpha\in(0,1]$.

(ii) $x_\alpha\in S(F(H))$, for some $\alpha\in(0,1]$ implies that $x\in S(H)$.

(iii) If $x_\alpha\in S(F(H))$, then $x_\beta\in S(F(H)),\ for\ all\ \beta\in(0,1]$.

(iv) $S(F(H))\subseteq F(\mu)$ if and only if $\mu(x)=1,\ for\ all\ x\in S(H)$.

(v) $F(\mu)\subseteq S(F(H))$ if and only if $F(H)=S(S(H))$.

Where $S(H)=\{x\in H\mid xox=\{0\}\}$.

Proof . (i) Let $x\in S(H)$. Then $xox=\{0\}$. So for all $\alpha\in(0,1]$, $x_\alpha\Theta x_\alpha=(xox)_\alpha=\{0_\alpha\}$. Therefore $x_\alpha\in S(F(H))$, , for all $\alpha\in(0,1]$.

(ii) Let $x_\alpha\in S(F(H))$ and $t\in xox$. Then $t_\alpha\in(xox)_\alpha=x_\alpha\Theta x_\alpha=\{0_\alpha\}$ .and so $t_\alpha=0_\alpha$

(iii) The proof follows from (i) and (ii).

(iv) Let $x \in S(H)$. Then by (i) $x_\alpha \in S(F(H))$, for all $\alpha \in (0,1]$. So $\mu(x) \geq \alpha$, for all $\alpha \in (0,1]$. Thus $\mu(x) = 1$

Conversely, let $x_\alpha \in S(F(H))$. Then $x \in S(H)$ by hypothesis $\mu(x) \geq \alpha$. So $x_\alpha \in F_\alpha(\mu) \subseteq F(\mu)$. Therefore $S(F(H)) \subseteq F(\mu)$.

(v) Let $x_\alpha \in F(H)$ and $\mu(x) = \beta$ Then $x_\beta \in F(\mu)$ implies that $x_\beta \in S(F(H))$. Thus by (iii) $x_\alpha \in S(F(H))$. So $F(H) \subseteq (F(H))$. Therefore F(H)=S(F(H)).

The converse is clear.

Proposition 3.16. Let μ be a fuzzy set of H. Then $F(\mu)$ is a quasi hyper BCK-subalgebra of F(H) if and only if $F_\alpha(\mu)$ is a fuzzy point hyper BCK-subalgebra of $F_\alpha(H)$, for all $\alpha \in (0,1]$..

Proof. Let $x_\alpha, y_\alpha \in F_\alpha(\mu)$. Then $x_\alpha, y_\alpha \in F(\mu)$. Since $F(\mu)$ is a quasi hyper BCK-subalgebra of F(H), then we get that $(xoy)_\alpha \subseteq x_\alpha \Theta y_\alpha = \bigcup_{\alpha \in (0,1]} F_\alpha(\mu)$.

Hence $x_\alpha \Theta y_\alpha \subseteq F_\alpha(\mu)$.

Conversely, let $x_{\alpha.}, x_\beta \in F(\mu)$. Then $\mu(x) \geq \alpha \geq \min\{\alpha, \beta\} = \gamma$ and $\mu(y) \geq \beta \geq \min\{\alpha, \beta\} = \gamma$ and so $x_\gamma, y_\gamma \in F_\gamma(\mu)$. . Thus $(xoy)_\gamma = x_\gamma \Theta y_\gamma \subseteq F_\gamma(\mu)$. Therefore $x_\alpha \Theta y_\beta = (xoy)_{\min\{\alpha,\beta\}=\gamma} \subseteq F_\gamma(\mu) \subseteq F(\mu)$, hence $F(\mu)$ is a quasi hyper BCK-subalgebra of F(H).

Theorem 3.17. Let μ be a fuzzy set in H such that $F(\mu)$ is a quasi hyper BCK - subalgebra of F(H). Then

(i) μ is a fuzzy hyper BCK-subalgebra of H,

(ii) $0_\alpha \in F(\mu)$ $for\ all\ \alpha \in \mathrm{Im}(\mu)$

Proof. (i) Follows from Theorem 3.12 and Proposition 3.16.

(ii) Let $\alpha \in \mathrm{Im}(\mu)$. Then there exists $x \in X$ such that $\mu(x) = \alpha$. Thus $x_\alpha \in F(\mu)$, and so $0_\alpha \in (xox)_\alpha = (x_\alpha \Theta x_a) \in F(\mu)$.

Proposition 3.18. If $F(\mu)$ is a quasi (weak, strong) hyper BCK-ideal of F(H), then $F_\alpha(\mu)$ is a fuzzy point (weak, strong) hyper BCK-ideal of $F_\alpha(H)$, for all $\alpha \in (0,1]$..

The following example shows that the converse of the above theorem is not true in general.

Example 3.19. In Example 3.5, we define fuzzy set μ on H by

$$\mu(0) = \mu(a) = 1/2 \qquad \mu(b) = 1/3$$

Then it is easy to check that $F_\alpha(\mu)$ is a fuzzy point (weak, strong) hyper BCK-ideal of $F_\alpha(H)$, for all $\alpha \in (0,1]$, but $F(\mu)$ is not a quasi (weak, strong) hyper BCK-ideal of $F(H)$, since $a_{2/3} \Theta b_{1/5} = (aob)_{1/5} \subseteq F(\mu)$ and $b_{1/5} \in F(\mu), but a_{2/3} \notin F(\mu)$

Theorem 3.20. μ is a fuzzy (strong) weak hyper BCK-ideal of H if and only if

(i) $0_\alpha \in F_\alpha(\mu),\ for\ all\ \alpha \in \mathrm{Im}(\mu),$

(ii) $((x_\alpha \Theta y_\beta) \cap F(\mu) \neq \emptyset)(x_\alpha \Theta y_\beta) \subseteq F(\mu)$ and $y_\beta \in F(\mu)$ imply that $x_{\min\{\alpha,\beta\}} \in F(\mu)$ for all $\alpha \in \mathrm{Im}(\mu)$

Proof . Let $(x_\alpha \Theta y_\beta) \subseteq F(\mu)\ and\ y_\beta \in F(\mu)$. Then $\mu(y) \geq \beta$ and $\inf_{a \in xoy} \mu(a) \geq \min\{\alpha, \beta\}$.

Thus by hypothesis we get that.

$$\mu(x) \geq \min\left\{\inf_{a \in xoy} \mu(a), \mu(y)\right\} \geq \min\{\alpha, \beta\}$$

Therefore $x_{\min\{\alpha,\beta\}} \in F(\mu)$. Also $\mu(0) \geq \mu(x),\ for\ all\ x \in H$ implies that $0_\alpha \in F(\mu)$ for all $all\ \alpha \in \mathrm{Im}(\mu)$.

Conversely, since $0_\alpha \in F(\mu)$ for all $\alpha \in \mathrm{Im}(\mu)$, then $\mu(0) \geq \mu(x)$, for all $x \in H$. Let $\min\left\{\inf_{\alpha \in xoy} \mu(a), \mu(y)\right\} = \alpha$. Then $\mu(y) \geq \alpha\ and\ \mu(a) \geq \alpha$ for all $a \in xoy$. Thus $x_\alpha \Theta y_\alpha = (xoy)_\alpha \subseteq F(\mu)$ and $y_\alpha \in F(\mu)$. Therefore $\mu(x) \geq \alpha = \min\left\{\inf_{a \in xoy} \mu(a), \mu(y)\right\}$ i.e μ is a fuzzy weak hyper BCK-ideal of H.

Remark. Since $x_\alpha \Theta y_\beta \prec F(\mu)$ for all $\alpha, \beta \in (0,1]$, the above theorem is not true for hyper BCK-ideal.

Definition 3.21. A non empty subset I of F(H) is called quasireflexive, if $x_\alpha \Theta x_\beta \subseteq I,\ for\ all\ \alpha, \beta \in (0,1]$.

Theorem 3.22. $I \subseteq F(H)$ is quasireflexive if and only if $x_\alpha \Theta x_\alpha \subseteq I$, for all $\alpha \in (0,1]$..

Proof. ($\rightarrow$) The proof is clear.

($\leftarrow$) By hypothesis we have

$$x_\alpha \Theta x_\beta = (xoy)_{\min\{\alpha,\beta\}} = x_{\min\{\alpha,\beta\}} \Theta x_{\min\{\alpha,\beta\}} \subseteq I.$$

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Proposition 3.23. Let μ be a fuzzy set. Then $F(\mu)$ is a quasireflexive if and only if $\mu(t)=1$ $for\ all\ t \in xox$.

Proof . $x_\alpha \Theta x_\alpha \subseteq F(\mu)$, for all $\alpha \in (0,1]$ implies that $\mu(t) \geq \alpha$, for all $\alpha \in (0,1]$ and $t \in xox$. Thus $\mu(t)=1$, for all $t \in xox$.

Conversely, the proof is easy.

Proposition 3.24. Let $I \subseteq F(H)$ be a quasireflexive. If I is a quasi hyper BCK-ideal, then it is a quasi strong hyper BCK-ideal.

Proof . Let $(x_\alpha \in y_\beta) \cap I \neq \emptyset$ and $y_\beta \in I$. Then there exists $t_{\min\{\alpha,\beta\}} \in (xoy)_{\min\{\alpha,\beta\}} \cap I.$ Now, let $s_{\min\{\alpha,\beta\}} \in x_\alpha \Theta y_\beta$. Then

$s_{\min\{\alpha,\beta\}} \Theta t_{\min\{\alpha,\beta\}} \subseteq (x_\alpha \Theta y_\beta)\Theta(x_\alpha \Theta y_\beta) \prec y_\beta \Theta y_\beta \subseteq I.$

So $t_{\min\{\alpha,\beta\}} \in I$ implies that $s_{\min\{\alpha,\beta\}} \in I$. Hence $x_\alpha \Theta y_\beta \prec I.$ Therefore by hypothesis we get that $x_\alpha \in I$.

We can prove that every strong quasi hyper BCK-ideal is a quasi hyper BCK-ideal, and so the converse of proposition 3.24 is true without any conditions.

I n F(H), consider the set.

$\Delta(a_t,b_s)=\{x_\alpha \in F(H) \mid 0_{\min\{\alpha,t,s\}} \in (x_\alpha \Theta a_t)\Theta b_s\}$

For all $a,b \in H$ and $\alpha,s,t \in (0,1]$. Obviously,

$0_\alpha, a_\alpha, b_\alpha \in \Delta(a_t,b_s),$ $for\ all\ \alpha \in (0,1]$ and

$\Delta(0_t,0_s)=\{0_\alpha \mid \alpha \in (0,1]\}$. Also

$(x_\alpha \Theta a_t)\Theta b_s = (x_\alpha \Theta b_s)\Theta a_t$ implies that

$\Delta(a_t,b_s)=\Delta(b_s,a_t)$, for all $a,b \in H$ and $s,t \in (0,1]$.

Let $a_t,b_s \in F(H)$. If there exists $u_\beta \in \Delta(a_t,b_s)$ such that $x_\alpha \prec u_\beta$ for all $x_\alpha \in \Delta(a_t,b_s)$, then we say that u_β the greatest quasi hyper element of $\Delta(a_t,b_s)$ and denoted such elements by $a_t \otimes b_s$.

If for all $a_t,b_s \in F(H), \Delta(a_t,b_s)$ has a greatest quasi hyper element, then the quasi hyper BCK-algebra F(H) is called to be with condition QH.

Example 3.25. Let $H = N \cup \{0\}$ and define hyperoperation “o” on H as follows:

$$xoy=\begin{cases}\{0,x\} & if\ x \leq y\\ \{x\} & if\ x > y\end{cases}$$

for all $x,y \in H$. Then (H, o, 0) is a hyper BCK- algebra. Let $a,b \in H$ and $x \leq a.$ Then

$(x_\alpha \Theta a_t)\Theta b_s = ((xoa)ob)_{\min\{\alpha,s,t\}} = (xob)_{\min\{\alpha,s,t\}} \cup \{0_{\min\{\alpha,s,t\}}\}$

So $0_{\min\{\alpha,s,t\}} \in (x_\alpha \Theta a_t)\Theta b_s$. Hence $x_\alpha \in \Delta(a_t,b_s)$, for all $\alpha \in [0,1]$.

Similarly, if $x \leq b$, then $x_\alpha \in \Delta(a_t,b_s)$. Now let x> a, x> b and b>a. Then

$(x_\alpha \Theta a_t)\Theta b_s = ((xoa)ob)_{\min\{\alpha,s,t\}} = (xob)_{\min\{\alpha,s,t\}}$. So by hypothesis that $0_{\min\{\alpha,s,t\}} \in (x_\alpha \Theta a_t)\Theta b_s$. Hence $x_\alpha \in \Delta(a_t,b_s)$. Therefore $\Delta(a_t,b_s)=F(H)$. So $\Delta(a_t,b_s)$ does not have the greatest quasi hyper element.

On the other hand

$$\begin{aligned}\Delta(a_t,0_s) &= \{x_\alpha \in F(H) \mid 0_{\min\{\alpha,t,s\}} \in (x_\alpha \Theta a_t)\Theta 0_s = ((xoa)o0)_{\min\{\alpha,t,s\}}\}\\ &= \{x_\alpha \in F(H) \mid 0_{\min(\alpha,t,s|} \in (xoa)_{\min\{\alpha,t,s\}}\}\\ &= \{x_\alpha \in F(H) \mid x \leq a\}\\ &= \bigcup_{a \in (0,1]} \{0,\ldots,a\}_\alpha\end{aligned}$$

Therefore $a_t \otimes 0_s = \{a_\alpha \mid \alpha \in (0,1]\}.$

In next proposition we will see that $a_t \otimes b_s$ has more than one element.

Proposition 3.26. Let F(H) be a quasi hyper BCK – algebra with condition QH. Then

(i) If u_γ is the greatest quasi hyper element of $\Delta(a_t,b_s)$, then for all $r \in (0,1]$, u_r is also the greatest quasi hyper element of $\Delta(a_t,b_s)$ and if v_s is another the greatest quasi hyper element of $\Delta(a_t,b_s)$, then $v_s = u_s,$

(ii) $a_t \otimes b_s = b_s \otimes a_t,$

(iii) $a_t \prec a_t \otimes b_s and$ $b_s \prec a_t \otimes b_s,$

(iv) $a_t \otimes 0_\alpha = \{a_\beta \mid \beta \in (0,1]\}$

(v) $\{u_r \mid r \in (0,1]\} = a_t \otimes b_s \subseteq F(\mu)$ if and only if $\mu(u)=1.$

Proof . (i) Let $c_\alpha \in \Delta(a_t,b_s)$ and $c_\alpha \prec u_\gamma.$ Then $0_{\min\{\alpha,\gamma\}} \in (cou)_{\min\{\alpha,\gamma\}}$ $and\ so\ 0 \in cou.$ Thus for all $r \in (0,1], 0_{\min\{\alpha,r\}} \in (cou)_{\min\{\alpha,t\}}$. Therefore $c_\alpha \prec u_r$. Also $u_\gamma \prec u_r, u_r \prec u_\gamma$ and $u_r \neq u_\gamma$ for all $\gamma \neq r$. Now let v_s be the greatest hyper element, then $u_s \prec v_s$ and $v_s \prec u_s$, so $u_s = v_s.$

(ii) the proof is easy.

(iii) $a_t,b_s \in \Delta(a_t,b_s)$ implies that $a_t \prec a_t \otimes b_s and$ $b_s \prec a_t \otimes b_s$. .

(iv) Let $x_\alpha \in \Delta(a_t, b_s)$. Then

$0_{\min\{\alpha,t,s\}} \in ((xoa)o0)_{\min\{\alpha,t,s\}}$ and so $0 \in xoa$. Hence

$0_{\min\{\alpha,t\}} \in (xoa)_{\min\{\alpha,t\}}$. Thus $x_\alpha \prec a_t$ and so

$a_t \otimes 0_\alpha = \{a_\beta \mid \beta \in (0,1]\}$.

(v) the proof is easy.

Conclusion

Quasi hyper BCK-algebra is a generalization of hyper BCK-algebras. We have introduced the concept of quasi hyper BCK-algebras and investigated some of their useful properties. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as hyper K -algebras, groups, semigroups, rings, nearrings, semirings (hemirings), lattices and Lie algebras. It is our hope that this work would other foundations for further study of the theory of BC K / BCI- algebras and hyper BCK-algebras. Our obtained results can be perhaps applied in engineering, soft computing or even in medical diagnosis.

In our future study of fuzzy structure of hyper BCK-algebras, may be the following topics should be considered:

(1) To get more connection to quasi hyper BCK-algebra and hyper BCK-algebra;

(2) To consider the structure of quotient hyper *BCK-algebras;*

(3) To get more results in quasi hyper *BCK-algebras* and application.

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