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Fuzzy BM-algebras

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Abstract

In this paper the notion of fuzzy BM-algebra and fuzzy topological BM-algebras are introduced. We stated and proved some theorem in fuzzy BM-algebras and level subalgebras. Finally fuzzy topological BM-algebras are studied.

Keywords: (fuzzy) BM-algebra, fuzzy BM-subalgebras, level subalgebras, fuzzy topological BM-algebras.
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Introduction

Imai and Iseki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras (Imai *et al.*, 1966; Iseki 1980). BCK-algebras is now known as a proper subclass of the class of BCI-algebras. HU & Li (1983; 1985) introduced a wide class of abstract algebras: BCH- algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers and Kim (1999) introduced the notion of d- algebras which is another generalization of BCK-algebras and also they introduced the notion of B-algebras (Neggers & Kim, 2002a,b). Moreover, Jun *et al.* (1998) introduced a new notion, called BH-algebra, which is a genrealization of BCH/BCI/BCK-algebras. Walendziak obtained other equivalent axioms for B-algebra (Walendziak, 2006). Kim *et al.* (Kim, 2004) introduced the notion a (pre-) Coxeter algebra and showed that it is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group. Kim and Kim (2006) introduced the notion of a BM-algebra which is a specialization of B-algebras.

The concept of a fuzzy set, which was introduced by Zadeh (1965), provides a natural framework for generalizing many of the concepts of general mathematics and topology. Foster (1979) combined the structure of a fuzzy topological spaces with that of a fuzzy goup, introduced by Rosenfeld (1971), to formulated the elements of a theory of fuzzy topological groups.

In the present paper, we introduced the concept of fuzzy BM-algebras and fuzzy topological BM-algebras and study this structure. We state and prove some theorem in fuzzy BM- subalgebras and level subalgebras. Finally, some of Fosters results on homomorphic images in fuzzy topological BM-algebras are studied.

Preliminary

Definition 2.1. (Kim & Kim, 2006): A BM-algebras is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

$(I)\ x*0=x,$

$(II)\ (z*x)*(z*y)=y*x,$

For all $x,y,z\in X$.

In X we can define a binary relation by $x\leq y$ if and only if $x*y=0$.

Definition 2.2. (Kim et al 2006) Let X be a BM-algebra. Then for x,y and z in X,

$(a)\ x*x=0,$

$(b)\,0*(0*x)=x,$

$(c)\,0*(x*y)=y*x,$

$(d)\,(x*z)*(y*z)=x*y,$

$(e)\,x*y=0$ if and only if $y*x=0,$

$(f)\,(x*y)*z=(x*z)*y.$

Definition 2.3. A non- empty subset S of a BM-algebra X is called a subalgebra of X if $x*y\in S$ for any $x,y\in S$.

A mapping $f:X\rightarrow Y$ of BM-algebras is called a BM-homomorphism if $f(x*y)=f(x)*f(y)$ for all $x,y\in X$.

We now review some fuzzy logic concept (Zadeh 1965).

Let X be a set. A fuzzy set A in X is characterized by a membership function $\mu_A:X\rightarrow[0,1]$. Let f be a mapping from the set X to the set Y and let B be a fuzzy set in Y with membership function μ_B.

The inverse image of B, denoted by $f^{-1}(B)$, is the fuzzy set in $\mu^{-1}_{f(B)}$ with membership function $\mu^{-1}_{f(B)}$ defined by $\mu_{f^{-1}(B)}(x)=\mu_B(f(x))$ for all $x\in X$.

Conversely, let A be a fuzzy set in X with membership function μ_A Then the image of, A, denoted by $f(A)$, is the fuzzy set in Y such that:

$$\mu_{f(A)}(y)=\begin{cases}\sup\limits_{z\in f^{-1}(y)} \mu_A(z) & if\ f^{-1}(y)\neq\phi,\\ 0 & otherwise,\end{cases}$$

A fuzzy set A in the BM-algebra X with the membership function μ_A is said to be have the sup property if for any subset $T\subseteq X$ there exists $x_0\in T$ such that

$$\mu_A(x_0)=\sup_{t\in T}\mu_A(t).$$

A fuzzy topology on a set X is a family τ of fuzzy sets in X which satisfy the following conditions:

(i) For $c\in[0,1], K_c\in\tau,$ where K_c has a constant membership function,

(ii) If $A,B\in\tau$, then $A\cap B\in\tau$,

(iii) If $A_j\in\tau$ for all $j\in J$, then $\bigcup\limits_{j\in J} A_j\in\tau$.

The pair (X,τ) is called a fuzzy topological space and members of τ are called open fuzzy sets.

Let A be a fuzzy set in X and τ a fuzzy topology on X. Then the induced fuzzy topology on A is the family of fuzzy subsets of A which are the intersection with A of τ-open fuzzy sets in X. The induced fuzzy topology is denoted by τ_A, and the pair (A,τ_A) is called a fuzzy subspace of (X,τ).

Let (X,τ) and (Y,v) are two fuzzy topological space. A mapping f of (X,τ) into (Y,v) is fuzzy continuous if for each open fuzzy set U in v the inverse image $f^{-1}(\mathrm{U})$ is in τ.

Conversely, f is fuzzy open if for each fuzzy set V in τ, the image $f(V)$ is in v.

Let (A,τ_A) and (B,v_B) are fuzzy subspace of fuzzy topological spaces (X,τ) and (Y,v) respectively, and let f be a mapping from (X,τ) to (Y,v).

Then f is a mapping of (A,τ_A) into (B,v_B) if $f(A)\subseteq B$. Furthermore f is relatively fuzzy continuous if for each open fuzzy set V' in v_B the intersection $f^{-1}(V')\cap A$ is in τ_A. Conversely, f is relatively fuzzy open if for each open fuzzy set U', the image $f(\mathrm{U}')$ is in v_B.

Lemma 2.4. (Foster 1979) Let $(A,\tau_A),(B,v_B)$ be fuzzy subspace of fuzzy topological spaces $(X,\tau),(Y,v)$ respectively, and let f be a fuzzy continuous mapping of (X,τ) into (Y,v) such that $f(A)\subset B$. Then f is a relatively fuzzy continuous mapping of (A,τ_A) into (B,v_B).

Fuzzy BM- subalgebra

From now on X is a BM-algebra, unless otherwise is stated.

Definition 3.1. Let μ be a fuzzy set in X. Then μ is called a fuzzy BM- subalgebra (algebra) of X if

$$\mu(x*y)\geq\min\{\mu(x),\mu(y)\}$$

for all $x,y\in X$.

Example 3.2. Let $X=\{0,1,2\}$ ba a set with the following table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X,*,0)$ is a BM- algebra.

Define a fuzzy set $\mu:X\to[0,1]$ by $\mu(0)=0.7>0.1=\mu(x)$ for all $x\in\{1,2\}$. Then μ is a fuzzy BM-subalgebra of X.

Lemma 3.3. If A is a fuzzy BM-subalgebra of X, then for all $x\in X$

$$\mu_A(0)\geq\mu_A(x).$$

Proof. For all $x\in X$, we have $x*x=0$ hence $\mu_A(0)=\mu_A(x*x)\geq\min\{\mu_A(x),\mu_A(x)\}=\mu_A(x).$

Proposition 3.4. Let A be a fuzzy BM-subalgebra of X, and let $n\in\mathrm{N}$.
Then

(i) $\mu_A\left(\prod\limits^{n} x*x\right)\geq\mu_A(x)$, for any odd number n,

(ii) $\mu_A\left(\prod\limits^{n} x*x\right)=\mu_A(x)$, for any even number n.

Proof: Let $x \in X$ and assume that n is odd. Then $n = 2k-1$ for some positive integer k. We prove by induction. Definition and above lemma imply that $\mu_A(x*x)=\mu_A(0)\ge\mu_A(x)$. Now suppose that $\mu_A\left(\prod^{2k-1} x*x\right)\ge\mu_A(x)$.

Then by assumption

$$\mu_A\left(\prod^{2(k+1)-1} x*x\right)=\mu_A\left(\prod^{2k+1} x*x\right)$$

$$=\mu_A\left(\prod^{2k-1} x*(x*(x*x))\right)$$

$$=\mu_A\left(\prod^{2k-1} x*x\right)\ge\mu_A(x)$$

Which proves (i). Similarly we can prove (ii).

Theorem 3.5. Let A be a fuzzy BM-subalgebra of X . If there exists a sequence $\{x_n\}$ in X , such that

$$\lim_{n\to\infty}\mu_A(x_n)=1$$

Then $\mu_A(0)=1$.

Proof. By above lemma we have $\mu_A(0)\ge\mu_A(x)$, for all $x\in X$, thus $\mu_A(0)\ge\mu_A(x_n)$, for every positive integer n . Consider

$$1\ge\mu_A(0)\ge\lim_{n\to\infty}\mu_A(x_n)=1.$$

Hence $\mu_A(0)=1$.

Theorem 3.6. Let A_1 and A_2 are fuzzy BM-subalgebras of X . Then $A_1\cap A_2$ is a fuzzy BM-subalgebras of X .

Proof. Let $x,y\in A_1\cap A_2$. Then $x,y\in A_1$ and A_2 , since A_1 and A_2 fuzzy BM-subalgebras of X by above theorem we have:

$$\mu_{A_1\cap A_2}(x*y)=\min\{\mu_{A_1}(x*y),\mu_{A_2}(x*y)\}$$

$$\ge\min\{\min(\mu_{A_1}(x),\mu_{A_1}(y)),\min(\mu_{A_2}(x),\mu_{A_2}(y))\}$$

$$=\min\{\mu_{A_1\cap A_2}(x),\mu_{A_1\cap A_2}(y)\}$$

Which proves theorem.

Corollary 3.7. Let $\{A_i \mid i\in\wedge\}$ be a family of fuzzy BM-subalgebras of X . Then $\bigcap_{i\in A} A_i$ is also an fuzzy BM-subalgebras of X .

Definition 3.8. Let A be a fuzzy set in X and $\lambda\in[0,1]$. Then the level BM-subalgebra $\mathrm{U}(A;\lambda)$ of X and strong level BM-subalgebra $\mathrm{U}(A;>,\lambda)$ of X are defined as following:

$$\mathrm{U}(A;\lambda):=\{x\in X \mid \mu_A(x)\ge\lambda\},$$

$$\mathrm{U}(A;>,\lambda):=\{x\in X \mid \mu_A(x)>\lambda\}$$

Theorem 3.9. Let A be a fuzzy BM-subalgebra of X with the least upper bound λ_0 . Then the following conditions are equivalent:

(i) A is a fuzzy BM- subalgebra of X .

(ii) For all $\lambda\in\mathrm{Im}(\mu_A)$, the nonempty level subset $\mathrm{U}(A;\lambda)$ of A is a BM- subalgebra of X .

(iii) For all $\lambda\in\mathrm{Im}(\mu_A)\setminus\lambda_0$, the nonempty strong level subset $\mathrm{U}(A:>,\lambda)$ of A is a BM-subalgebra of X .

(iv) For all $\lambda\in[0,1]$, the nonempty strong level subset $\mathrm{U}(A,>,\lambda)$ of A is a BM-subalgebra of X .

(v) For all $\lambda\in[0,1]$, the nonempty level subset $\mathrm{U}(A;\lambda)$ of A is a BM-subalgebra of X .

Proof. $(i\to iv)$ Let A be a fuzzy BM- subalgebra of $X, \lambda\in[0,1]$ and $x,y\in U(A;<,\lambda)$, then we have

$$\mu_A(x*y)\ge\min\{\mu_A(x),\mu_A(y)\}>\min\{\lambda,\lambda\}=\lambda$$

Thus $x*y\in\mathrm{U}(A;>,\lambda)$. Hence $\mathrm{U}(A;\lambda)$ is a BM-subalgebra of X .

$(iv\to iii)$ It is clear.

$(iii\to ii)$ Let $\lambda\in\mathrm{Im}(\mu_A)$. Then $\mathrm{U}(A,\lambda)$ is nonempty. Since $\mathrm{U}(A,\lambda)=\bigcap_{\lambda>\beta}\mathrm{U}(A;>,\lambda)$, where $\beta\in\mathrm{Im}\,\mu_A\setminus\lambda_0$. Then by (iii) and Corollary 3.7, $\mathrm{U}(A,\lambda)$ is a BM-subalgebra of X .

$(ii\to v)$ Let $\lambda\in[0,1]$ and $\mathrm{U}(A;\lambda)$ be nonempty. Suppose $x,y\in\mathrm{U}(A;\lambda)$ .Let $\alpha=\min\{\mu_A(x),\mu_A(y)\}$, it is clear that

$\alpha=\min\{\mu_A(x),\mu_A(y)\}\geq\{\lambda,\lambda\}=\lambda$. Thus $x,y\in \mathrm{U}(A;\alpha)$ and $\alpha\in\mathrm{Im}(\mu_A)$, by (ii) $\mathrm{U}(A;\alpha)$ is a BM-subalgebra of X, hence $x*y\in\mathrm{U}(A;\alpha)$. Then we have

$$\mu_A(x*y)\geq\min\{\mu_A(x),\mu_A(y)\}\geq\{\alpha,\alpha\}=\alpha\geq\lambda.$$

Therefore $x*y\in\mathrm{U}(A;\lambda)$. Then $\mathrm{U}(A;\lambda)$ is a BM-subalgebra of X.

$(v\to i)$ Assume that the nonempty set $\mathrm{U}(A;\lambda)$ is a BM-subalgebra of X, for every $\lambda\in[0,1]$. In contrary, let $x_0,y_0\in X$ be such that

$$\mu_A(x_0*y_0)<\min\{\mu_A(x_0),\mu_A(y_0)\}.$$

Let $\mu_A(x_0)=\gamma,\mu_A(y_0)=\theta$ and $\mu_A(x_0*y_0)=\lambda$. Then

$$\lambda<\min\{\gamma,\theta\}$$

Consider

$$\lambda_1=\frac{1}{2}\left(\mu_A(x_0*y_0)+\min\{\mu_A(x_0),\mu_A(y_0)\}\right)$$

We get that

$$\lambda_1=\frac{1}{2}\left(\lambda+\min\{\gamma,\theta\}\right)$$

Therefore

$$\gamma>\lambda_1=\frac{1}{2}\left(\lambda+\min\{\gamma,\theta\}\right)>\lambda$$

$$\theta>\lambda_1=\frac{1}{2}\left(\lambda+\min\{\gamma,\theta\}\right)>\lambda$$

Hence

$$\min\{\gamma,\theta\}>\lambda_1>\lambda=\mu_A(x_0*y_0)$$

so that $x_0*y_0\notin\mathrm{U}(A;\lambda)$. Which is a contradiction, since

$$\mu_A(x_0)=\gamma\geq\min\{\gamma,\theta\}>\lambda_1$$

$$\mu_A(y_0)=\theta\geq\min\{\gamma,\theta\}>\lambda_1$$

imply that $x_0,y_0\in\mathrm{U}(A;\lambda)$. Thus $\mu_A(x*y)\geq\min\{\mu_A(x),\mu_A(y)\}$ for all $x,y\in X$. Which completes the proof.

Theorem 3.10. Each BM-subalgebra of X is a level BM-subalgebra of a fuzzy BM-subalgebra of X.

Proof. Let Y be a BM-subalgebra of X, and A be an fuzzy set on X defined by

$$\mu_A(x)=\begin{cases}\alpha & \text{if } x\in Y\\ 0 & \text{otherwise,}\end{cases}$$

where $\alpha\in[0,1]$. It is clear that $\mathrm{U}(A;\alpha)=Y$. Let $x,y\in X$. We consider the following cases:

Case 1) If $x,y\in Y$, then $x*y\in Y$ therefore $\mu_A(x*y)=\alpha=\min\{\alpha,\alpha\}=\min\{\mu_A(x),\mu_A(y)\}$.

Case 2) If $x,y\notin Y$, then $\mu_A(x)=0=\mu_A(y)$ and so $\mu_A(x*y)\geq 0=\min\{0,0\}=\min\{\mu_A(x),\mu_A(y)\}$.

Case 3) If $x\in Y$ and $y\notin Y$, then $\mu_A(x)=\alpha$ and $\mu_A(y)=0$. Thus

$$\mu_A(x*y)\geq 0=\min(\alpha,0)=\min\{\mu_A(x),\mu_A(y)\}.$$

Case 4) If $y\in Y$ and $x\notin Y$, then by the same argument as in case 3, we can conclude that $\mu_A(x*y)\geq\min\{\mu_A(x),\mu_A(y)\}$ .Therefore A is a fuzzy BM-subalgebra of X.

In the next theorem we generalize the above theorem.

Theorem 3.11. Let X be a BM-algebra. Then for any chain of subalgebras

$$A_0\subset A_1\subset\ldots\subset A_r=X$$

there exists a fuzzy subalgebra μ of X whose level subalgebras are exactly the subalgebras of this chain.

Proof. Consider a set of numbers

$$t_0>t_1>\ldots>t_r$$

Where each t_i be in $[0,1]$. Define $\mu:X\to[0,1]$ by $\mu(A_i\setminus A_{i-1})=t_i$ for all $0<i\leq r$ and $\mu(A_0)=t_0$.

We prove that μ is a fuzzy subalgebra of X. Let $x,y\in X$, We consider the following cases:

Case 1) Let $x,y\in A_i\setminus A_{i-1}$, then $\mu(x)=t_i=\mu(y)$. Since A_i is a subalgebra thus $x*y\in A_i$, so $x*y\in A_i\setminus A_{i-1}$ or $x*y\in A_{i-1,}$, and in each of them we have

$$\mu(x*y)\geq t_i=\min\{\mu(x),\mu(y)\}$$

Case 2) Let $x\in A_i\setminus A_{i-1,}$ $y\in A_i\setminus A_{j-1},$ where $i<j$. Then $\mu(x)=t_i$ $\mu(y)=t_{j,}$ since $A_i\subseteq A_j$ and A_j is a subalgebra of X, then $x*y\in A_j$.

Hence

$$\mu(x*y)\geq t_j=\min\{\mu(x),\mu(y)\}.$$

It is clear that $\mathrm{Im}(\mu)=\{t_{0,}t_1,...,t_r\}$, therefore the level subalgebras of μ are given by the chain of subalgebras

$$\mu_{t_0}\subset\mu_{t_1}...\subset\mu_{t_r}=X$$

We have $\mu_{t0}=\{x\in X \mid \mu(x)\geq t_0\}=A_0$. It is clear that $A_i\subseteq\mu_{t_i}$. Let $x\in\mu_{t_i}$ Then $\mu(x)\geq t_i$ thus $x\notin A_j$ for $j>i$. So $\mu(x)\in\{t_0,t_1,...,t_i\}$, thus $x\in A_k$, for $\mathrm{k}\leq\mathrm{i}$, since $A_k\subseteq A_i$ we get that $x\in A_i$. Hence $A_i=\mu_{t_i}$ for $0\leq i\leq r$.

Theorem 3.12. Let X be a BM-algebra. Then two level subalgebras μ_{t_1},μ_{t_2} (where $t_1<t_2$) of μ are equal if and only if there is no $x\in X$ such that $t_1\leq\mu(x)<t_2$.

Proof. In contrary let $\mu_{t_1}=\mu_{t_2}$ where $t_1<t_2$ and there exists $x\in X$ such that $t_1\leq\mu(x)<t_2$. Then μ_{t_2} is a proper subset of μ_{t_1} which is a contradiction.

Conversely, suppose that there is no $\mathrm{x}\in X$ such that $t_1\leq\mu(x)<t_2$. Since $t_1<t_2$ then $\mu_{t_1}\subseteq\mu_{t_2}$. If $\mathrm{x}\in\mu_{t_2}$ then $\mu(x)\geq t_1$ by hypotheses we get that $\mu(x)\geq t_2$. Therefore $x\in\mu_{t_2}$ then $\mu_{t_2}\subseteq\mu_{t_1}$ Hence $\mu_{t_1}=\mu_{t_2}$.

Theorem 3.13. Let Y be a subset of X and A be a fuzzy set on X which is given in the proof of Theorem 3.10. If A is a fuzzy BM-subalgebra of X, then Y is a BM-subalgebra of X.

Proof. Let A be a fuzzy BM-subalgebra of X, and $\mathrm{x},\mathrm{y}\in Y$. Then $\mu_A(x)=\alpha=\mu_A(y)$, thus

$$\mu_A(x*y)\geq\min\{\mu_A(x),\mu_A(y)\}=\min\{\alpha,\alpha\}=\alpha.$$

Which implies that $x*y\in Y$.

Theorem 3.14. If A is a fuzzy BM-subalgebra of X, then the set

$$X_{\mu_A}:=\{x\in X \mid \mu_A(x)=\mu_A(0)\}$$

is a BM-algebra of X.

Proof. Let $x,y\in X_{\mu_A}$. Then $\mu_A(x)=\mu_A(0)=\mu_A(y)$, and so

$$\mu_A(x*y)\geq\min\{\mu_A(x),\mu_A(y)\}=$$
$$\min\{\mu_A(0),\mu_A(0)\}=\mu_A(0)$$

by lemma 3.3, we get that $\mu_A(x*y)=\mu_A(0)$ which means that $x*y\in X_{\mu_A}$.

Theorem 3.15. Let N be a fuzzy subset of X. Let N be a fuzzy set defined by μ_N as:

$$\mu_N(x)=\begin{cases}\alpha & \text{if } x\in N,\\ \beta & \text{otherwise,}\end{cases}$$

For all $\alpha,\beta\in[0,1]$ with $\alpha\geq\beta$. Then N is a fuzzy BM-subalgebra if and only if N is a BM-subalgebra of X. Moreover, in this case $X_{\mu_N}=N$.

Proof. Let N be fuzzy a BM-subalgebra. Let $x,y\in X$ be such that $\mathrm{x},\mathrm{y}\in N$ Then

$$\mu_N(x*y)\geq\min\{\mu_N(x),\mu_N(y)\}=\min(\alpha,\alpha)=\alpha$$

and so $x*y\in N$.

Conversely, suppose that N is a BM- subalgebra of X, let $x,y\in X$.

(i) If $x,y\in N$ then $x*y\in N$, thus

$$\mu_N(x*y)=\alpha=\min\{\mu_N(x),\mu_N(y)\}$$

(ii) If $x\notin N$, or $y\notin N$, then

$$\mu_N(x*y)\geq\beta=\min\{\mu_N(x),\mu_N(y)\}$$

This shows that N is a fuzzy BM-subalgebra. Moreover, we have

$$X_{\mu_N}:=\{x\in X \mid \mu_N(x)=\mu_N(0)\}=\{x\in X \mid \mu_N(x)=\alpha\}=N$$

Fuzzy topological BM-algebra

Proposition 4.1. Let f be a BM- homomorphism from X into Y and G be a fuzzy BM-algebra of Y with the membership function μ_G. Then the inverse image $f^{-1}(G)$ of G is a fuzzy BM-algebra of X.

Proof. Let $x,y\in X$. Then

$$\mu_{f^{-1}(G)}(x*y)=\mu_G(f(x*y))$$
$$=\mu_G(f(x)*f(y))$$
$$\geq\min\{\mu_G(f(x)),\mu_G(f(y))\}$$
$$=\min\{\mu_{f^{-1}(G)}(x),\mu_{f^{-1}(G)}(y)\}$$

Proposition 4.2. Let f be a BM-homomorphism from X onto Y and D be a fuzzy BM-algebra of X with the sup property. Then the image $f(D)$ of D is a fuzzy BM-algebra of Y.

Proof. Let $a, b \in Y$, and $x_0 \in f^{-1}(a), y_0 \in f^{-1}(b)$ such that

$$\mu_D(x_0) = \sup_{t \in f^{-1}(a)} \mu_D(t),\ \mu_D(y_0) = \sup_{t \in f^{-1}(b)} \mu_D(t).$$

Then by the definition of $\mu_{f(D)}$, we have

$$\mu_{f(D)}(x*y) = \sup_{t \in f^{-1}(a*b)} \mu_D(t)$$
$$\geq \mu_D(x_0 * y_0)$$
$$\geq \min\{\mu_D(x_0), \mu_D(y_0)\}$$
$$= \min\left\{\sup_{t \in f^{-1}(a)} \mu_D(t), \sup_{t \in f^{-1}(b)} \mu_D(t)\right\}$$
$$= \min\left\{\mu_{f(D)}(a), \mu_{f(D)}(b)\right\}.$$

For any BM-algebra X and any element $a \in X$ we denote by R_a the right translation of X defined by $R_a(x) = x*a$ for all $x \in X$. It is clear that $R_0(x) = 0 = R_x(x)$.for all $x \in X$

Definition 4.3. Let τ be a fuzzy topology on X and D be a fuzzy BM-algebra of X with induced topology τ_D. Then D is called a fuzzy topological BM-algebra of X if for each $a \in X$ the mapping $R_a : (D, \tau_D) \to (D, \tau_D)$ is relatively fuzzy continuous.

Theorem 4.4. Let X and Y be two BM-algebras, $f : X \to Y$ be a BM-homomorphism. Let τ and v be the fuzzy topologies on X and Y respectively, such that $\tau = f^{-1}(v)$. Let G be a fuzzy topological BM-algebra of μ with membership function μ_G. Then $f^{-1}(G)$ is a fuzzy topological BM-algebra of X with membership function $\mu_{f^{-1}(G)}$.

Proof. We must show that, for each $a \in$, the mapping

$$R_a : \left(f^{-1}(G), \tau_{f^{-1}(G)}\right) \to \left(f^{-1}(G), \tau_{f^{-1}(G)}\right)$$

is relatively fuzzy continuous. Let U be any open fuzzy set in $\tau_{f^{-1}(G)}$ on $f^{-1}(G)$.

Since f is a fuzzy continuous mapping from (X, τ) into (Y, v), from Lemma 2.4 follows that f is a relatively fuzzy continuous mapping of $\left(f^{-1}(G), \tau_{f^{-1}(G)}\right)$ into (G, v_G). Note that there exists an open fuzzy set V such that $f^{-1}(V) = \mathrm{U}$. The membership function of $R_a^{-1}(\mathrm{U})$ is given by

$$\mu_{R_a^{-1}(\mathrm{U})}(x) = \mu_{\mathrm{U}}(R_a(x)) = \mu_{\mathrm{U}}(x*a)$$
$$\mu_{f^{-1}(V)}(x*a) = \mu_V(f(x*a))$$
$$= \mu_V(f(x)*f(a))$$

Since G is a fuzzy topological BM-algebra of Y, the mapping

$$R_b : (G, v_G) \to (G, v_G)$$

is relatively fuzzy continuous for each $b \in Y$. Hence

$$\mu_{R_a^{-1}(\mathrm{U})}(x) = \mu_V(f(x)*f(a)) = \mu_{\mathrm{v}}\left(R_{f(a)}(f(x))\right)$$

$$= \mu_{R^{-1}_{f(a)}(V)}(f(x)) = \mu_{f^{-1}\left(R^{-1}_{f(a)}(V)\right)}(x).$$

Which implies that $R_a^{-1}(\mathrm{U}) = f^{-1}\left(R^{-1}_{f(a)}(V)\right)$ therefore

$$R_a^{-1}(\mathrm{U}) \cap f^{-1}(G) = f^{-1}\left(R^{-1}_{f(a)}(V)\right) \cap f^{-1}(G)$$

is open in the relative fuzzy topology on $f^{-1}(G)$.

The membership function μ_G of fuzzy BM-algebra G of X is said to be f-invariant if $f(x) = f(y)$ implies $\mu_G(x) = \mu_G(y)$, for all $x, y \in X$.

Theorem 4.5. Given BM-algebras X and Y and a BM-homomorphism f from X onto Y, let τ be the fuzzy topology on X and v be the fuzzy topology on Y such that $f(\tau) = v$. Let D be a fuzzy topological BM-algebra of X. If the membership function μ_D of D is a f-invariant, then $f(D)$ is a fuzzy topological BM-algebra of Y.

Proof. It is enough to show that the mapping

$$R_b : \left(f(D), v_{f(D)}\right) \to \left(f(D), v_{f(D)}\right)$$

Is relatively fuzzy continuous, for all $b \in Y$. It is clear that f is a relatively fuzzy open mapping, since for $\mathrm{U} \in \tau_D$ there exists $\mathrm{U}' \in \tau$ such $\mathrm{U} = \mathrm{U}' \cap D$,by f-invariance of μ_D we have

$$f(\mathrm{U}) = f(\mathrm{U}) \cap f(D) \in v_{f(D)}.$$

Let V' be an open fuzzy set in $v_{f(D)}$. For any $b \in Y$ by hypothesis there exists $a \in X$ such that $b = f(a)$. Thus

$$\mu_{f^{-1}\left(R_b^{-1}(V')\right)}(x)$$

$$= \mu_{f^{-1}\left(R_{f(a)}^{-1}(V')\right)}(x) = \mu_{R_{f(a)}^{-1}(V')}\left(f(x)\right)$$

$$= \mu_{V'}\left(R_{f(a)}\left(f(x)\right)\right) = \mu_{V'}\left(f(x) * \left(f(a)\right)\right)$$

$$= \mu_{V'}\left(f(x * a)\right)$$

$$= \mu_{f^{-1}(V')}(x * a)$$

$$= \mu_{f^{-1}(V')}\left(R_a(x)\right) = \mu_{R_a^{-1}\left(f^{-1}(V')\right)}(x)$$

Which implies that $f^{-1}\left(R_b^{-1}(V')\right) = R_a^{-1}\left(f^{-1}(V')\right)$. By hypothesis, R_a is a relatively fuzzy continuous mapping from (D, τ_D) to (D, τ_D) and f is a relatively fuzzy continuous mapping from (D, τ_D) to $\left(f(D), v_{f(D)}\right)$. Therefore

$$f^{-1}\left(R_b^{-1}(V')\right) \cap G = R_a^{-1}\left(f^{-1}(V')\right) \cap D$$

is open in τ_D. Since f is relatively fuzzy open, then

$$f\left(f^{-1}\left(R_b^{-1}(V')\right) \cap D\right) = R_b^{-1}(V') \cap f(D)$$

is open in $v_{f(D)}$.

Conclusion and future research

In this paper, we introduced the notion of the fuzzy BM-algebras and we studied this structure. In addition, we give relationship between fuzzy BM-algebra and level subsets. Finally, we gave some of Fosters results on homomorphic image and inverse image in fuzzy topological BM- algebras.

In our future work, we are going to consider the relationship between this fuzzy structure and the other types of fuzzy structure.

We hope this work would serve as a foundation for further studies on the structure of fuzzy BM-algebras and develop corresponding many-valued logical systems.

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