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An investigation on two-dimensional non-linear wave equation using VIM

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Abstract

In this article, a kind of analytical method called He's variational iteration method (VIM) have been used to obtain the analytical solution of two-dimensional nonlinear wave equation. In this method, general Lagrange multiplier is introduced to construct correction functions for the problems. The multiplier can identify optimally via the variational theory. The results compare with those of exact solutions and homotopy perturbation method. A clear conclusion can be drawn from the numerical results and the proposed method provides excellent approximations to the solution of this kind of nonlinear wave phenomenon in terms of simplicity and accuracy. Thus, it can be easily extended to other nonlinear wave phenomena finding wide application.

Keywords: Variational iteration method, wave equation.

Introduction

Wave phenomena are observed in fluid dynamics, plasma, elastic media, optical fibers, etc. In the past, both mathematicians and physicists have made significant progress in this direction (Khatami *et al.*, 2008; Mahmoudi *et al.*, 2008). In most cases, the wave phenomena are based on the functional equations. Except a limited number of these problems, most of them function nonlinear thus, do not have precise analytical solutions. On the other hand, solving these nonlinear equations analytically may guide authors to know the described process deeply. For this purpose, many different methods have been tried recently; for instance, the homotopy analysis method (Abbasbandy, 2006), the variational iteration method (VIM) (Ganji et al, 2006; Sadighi & Ganji, 2007; He, 2000), the Adomian's decomposition method (ADM) (Adomian, 1994; Wazwaz, 2007), homotopy perturbation method (Tolou *et al.*, 2008) and Exp-function method.

This paper deals with analytical solution of two-dimensional nonlinear wave equation by the means of VIM. The results have been comparing with those of exact solution and homotopy perturbation method (HPM) of (Ghasemi *et al.*, 2007). The results show the convergence to correct results by VIM, is better and faster than HPM.

Materials and Methods

Basic idea of He's variational iteration method

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t) \qquad (1)$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is an inhomogeneous term.

According to VIM, we can write down a correction function as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\right)d\tau \qquad (2)$$

where λ is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript n indicates the n^{th} approximation and $\tilde{u}_n$ has considered as a restricted variation $\delta\tilde{u}_n = 0$.

Implementation of VIM to wave equation

To give a clear overview of the methodology, as an illustrative example, we take two-dimensional nonlinear wave equation described by:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - u\dfrac{\partial^2 u}{\partial t^2} = 1 - \dfrac{x^2+t^2}{2}, & 0 \le x,t \le 1, \\ u(0,t) = \dfrac{t^2}{2}, & \dfrac{\partial}{\partial x}u(0,t) = 0, \end{cases} \qquad (3)$$

Using Eqs. (1) and (2), the correction functional obtained as:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(\tau)\left\{u_n'' - u_n\ddot{u}_n - 1 + \frac{x^2+t^2}{2}\right\}d\tau, \qquad (4)$$

where prime indicates a differential with respect to *t* and dot denotes a differential with respect to *x*. λ is general Lagrangian multiplier.

After some manipulation, the following stationary conditions have been obtained:

$$\lambda'(\tau) = 1, \qquad (5)$$

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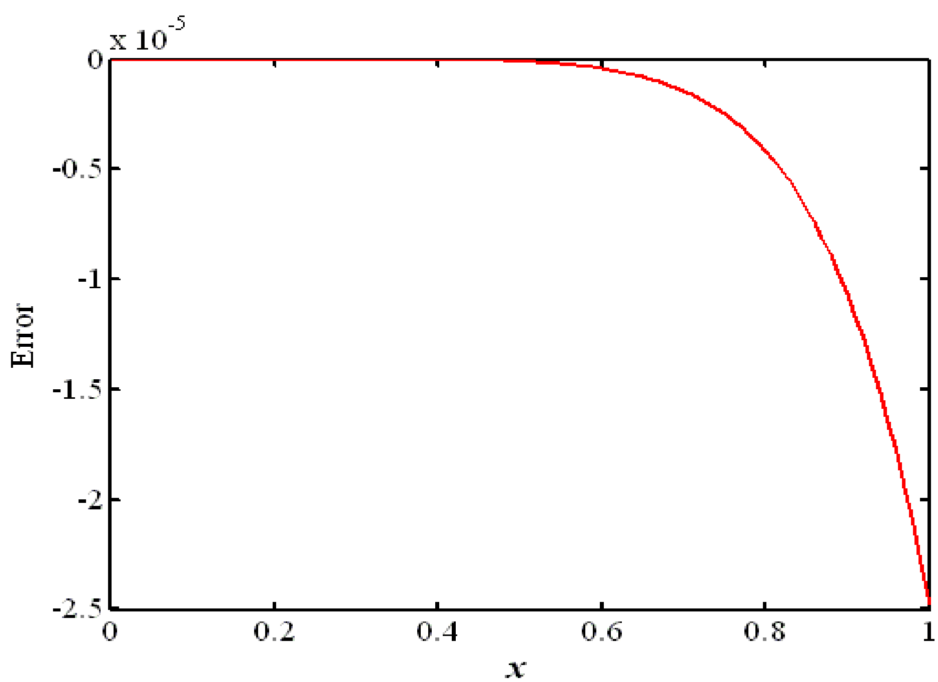
Table 1. Comparison of the results obtained by VIM, HPM to the exact solution

x	*t*	Exact value	VIM value	Error $(u_{VIM} - u_{Exact})$	Error $(u_{HPM} - u_{Exact})$
0	0	0	0	0	0
0.062	0.162	0.0150440	0.015043999999995	$-5.0000000000\times10^{-15}$	6.003×10^{-12}
0.156	0.156	0.0243360	0.024335999991301	$-8.6990000000\times10^{-12}$	1.128×10^{-9}
0.187	0.275	0.0552970	0.055296999962914	$-3.7086000000\times10^{-11}$	1.213×10^{-8}
0.225	0.128	0.0335045	0.033504499837093	$-1.6290700000\times10^{-10}$	2.613×10^{-9}
0.281	0.281	0.0789610	0.078960999035882	$-9.6411800000\times10^{-10}$	1.275×10^{-7}
0.343	0.343	0.1176490	0.117648995248481	$-4.7515190000\times10^{-9}$	6.370×10^{-7}
0.350	0.263	0.0958345	0.095834494414992	$-5.5850080000\times10^{-9}$	3.309×10^{-7}
0.350	0.870	0.4397000	0.439699994414992	$-5.5850080000\times10^{-9}$	5.736×10^{-6}
0.369	0.384	0.1418085	0.141808491475078	$-8.5249220000\times10^{-9}$	1.273×10^{-6}
0.468	0.468	0.2190240	0.219023942924950	$-5.7075050000\times10^{-8}$	7.922×10^{-7}
0.539	0.403	0.2264650	0.226464823319102	$-1.7668089800\times10^{-7}$	1.122×10^{-5}
0.593	0.593	0.3516490	0.351648620757480	$-3.7924252000\times10^{-7}$	5.497×10^{-5}
0.620	0.685	0.4268125	0.426811958481881	$-5.4151811900\times10^{-7}$	1.010×10^{-4}
0.656	0.656	0.4303360	0.430335149429744	$-8.5057025600\times10^{-7}$	1.263×10^{-4}
0.780	0.111	0.3103605	0.310357101898882	$-3.3981011180\times10^{-6}$	9.783×10^{-5}
0.781	0.781	0.6099610	0.609957566889773	$-3.4331102270\times10^{-6}$	5.378×10^{-4}
0.843	0.843	0.7106490	0.710642674424065	$-6.3255759350\times10^{-6}$	1.019×10^{-3}
0.968	0.968	0.9370240	0.937004880244468	$-1.9119755532\times10^{-5}$	3.271×10^{-3}
0.975	0.692	0.7147445	0.714724245738996	$-2.0254261003\times10^{-5}$	1.461×10^{-3}
1.000	1.000	1.0000000	0.999975198412698	$-2.4801587302\times10^{-5}$	4.310×10^{-3}

Fig.1. Error value



$$1+\lambda(\tau)\big|_{\tau=x}=0, \quad (6)$$

The Lagrangian multipliers can be identified as:

$$\lambda=(\tau-x), \quad (7)$$

Substituting Eq. (7) into the correction functional equation system (4), results in the following iteration formula:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^x (x-\tau)\left\{u_n''-u_n\ddot{u}_n-1+\frac{x^2+t^2}{2}\right\}d\tau, \quad (8)$$

Each result obtained from Eq. (8) is $u(x,t)$ with its own error relative to the exact solution, but higher number iterations leads us to obtain results closer to the exact solution. Using the iteration formula (8) and the initial condition as $u_0(x,t)$, two iterations has made as follows:

$$u_1(x,t)=\frac{t^2+x^2}{2}-\frac{1}{24}x^4, \quad (9)$$

$$u_2(x,t)=\frac{t^2+x^2}{2}-\frac{1}{720}x^6, \quad (10)$$

If we continue iterations, we can obtain:

$$u_3(x,t)=\frac{t^2+x^2}{2}-\frac{1}{40320}x^8, \quad (11)$$

$$u_4(x,t)=\frac{t^2+x^2}{2}-\frac{1}{3628800}x^{10}, \quad (12)$$

$$u_5(x,t)=\frac{t^2+x^2}{2}-\frac{1}{479001600}x^{12}, \quad (13)$$

As we can see with continue of iterations, the last sentences leading to zero. Therefore, $u(x,t)$ can leads to following exact formula:

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=\frac{t^2+x^2}{2}, \quad (14)$$

In (Ghasemi *et al.*, 2007) Eq. (3) solved by HPM for two iterative and result obtained as following:

$$u_{HPM}=\frac{x^2+t^2}{2}+\frac{1}{2688}x^8+\frac{1}{240}x^6t^2-\frac{327}{241920}x^8 +\frac{1}{1120}x^8t^2-\frac{1}{21600}x^{10}t^2+\frac{1038}{368800}x^{10}-\frac{1}{107520}x^{12}, \quad (15)$$

Results and discussions

In order to provide an evident of accuracy and efficiency of proposed solution, the result obtained by VIM and HPM (Ghasemi et al., 2007), both for two iterations have been compared with exact solution in the form of $(u_{VIM}-u_{Exact})$ and $(u_{HPM}-u_{Exact})$, as indicated in Table1.

Table 1 is clearly confirms that the numerical results of approximate solution for discussed problem from VIM are in excellent agreement with those of exact solution and HPM. As a mater of fact, the results obtained by VIM are clearly more convergence and accrue than HPM. For further verification, the $(u_{VIM}-u_{Exact})$ has been portrait graphically in Fig.1. This figure illustrates during the initial time thus, for the limited range of *x*, the difference between the analytical (VIM) and numerical solutions is negligible and the maximum value of error in this range is less than -2.5×10^{-5}.

Conclusion

In this survey, He's variational iteration method has been successfully applied to find the solution of two-dimensional nonlinear wave equation. Results of the present method are in excellent agreement with those of exact solution. Using VIM, the initial solution can be freely chosen with some unknown parameters. Moreover, in the proposed method, the unknown parameters in the initial solution can easily achieved while it is quite uncommon in other methods. Also, it can be concluded from VIM, fewest number of iterations or even in some cases once is sufficient to achieve converge to correct results in comparison to other methods.

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