

(a,d)-Antimagic Labelings in Extended Duplicate Twig Graphs

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Abstract

In this paper, we show that the class of Extended Duplicate Graphs of a Twig is (a,d)-edge-antimagic vertex labeling.
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Introduction

The concept of graph labeling was introduced by Rosa in 1967. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs serve as useful models for a broad range of applications such as circuit design, radar, astronomy, coding theory, communication network addressing and data base management (Gallian, 2010). Hence in the intervening years various labeling of graphs such as graceful labeling, harmonious labeling, magic labeling, antimagic labeling, bimagic labeling, prime labeling, cordial labeling, mean labeling, arithmetic labeling etc., have been studied in over 1100 papers (Gallian, 2010).

Hartsfield and Ringel (1990) introduced the concept of an antimagic graphs in 1990. In their terminology, a graph $G(V,E)$ is called antimagic if its edges are labeled with labels $1,2,\dots,|E(G)|$ in such a way that all vertex-weights are pair wise distinct, where a vertex-weight v is the sum of labels of all edges incident with v . They conjectured that every tree except K_2 is antimagic and moreover the connected graphs except K_2 is antimagic.

The concept of (a,d) - antimagic labeling was introduced by Bodendiek and Walther (1997). They proved that even cycles, paths of even order are not (a,d)-antimagic. They also proved a tree of odd order $2k+1$ ($k > 1$) is (a,d)-antimagic. Yegnanarayanan (2001) introduced several variations of antimagic labelings and established some results. Nicholas *et al.* (2004) has proved $K_{m,n}$ is (a,d) antimagic where $m \leq n$ and also that

$K_{m,n}$ is not (a,d) antimagic where $n > m > 1$. In this paper, we prove that the Extended Duplicate Graph of Twigs is (a,d)-edge-antimagic vertex labeling.

Preliminaries

In this section we give the basic notions relevant to this paper. Let $G = G(V, E)$ be a finite, simple and undirected graph with p vertices and q edges. By a labeling we mean a one-to-one mapping that carries a set of graph elements onto a set of numbers, called labels.

In this paper we deal labeling with domain as the set of all vertices.

Definition ((a,d)-edge-antimagic vertex labeling): A function f is said to be (a,d)-edge-antimagic vertex labeling of a graph G with q edges if f is 1-1 from $V \rightarrow \{1,2,3,\dots,|V|\}$ such that for each edge uv assigned the label $\{f(u) + f(v) ; uv \in E\}$, the resulting edge labels are distinct numbers $\{a, a+d, a+2d, \dots, a+(|E|-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

Definition (Twig): A graph $G(V, E)$ obtained from a path by attached exactly two pendent edges to each internal vertices of the path is called a Twig graph. Generally, a Twig T_m with m internal vertices; has $3m+1$ edges and $3m+2$ vertices. We classify the Twigs into two categories such as T_{2n} and T_{2n-1} , $n \in \mathbb{N}$.

Definition: Let $G(V,E)$ be a simple graph. A Duplicate graph of G is $DG = (V_1, E_1)$ where the vertex set $V_1 = V \cup V'$ and $V \cap V' = \phi$ and $f: V \rightarrow V'$ is bijective, (for $v \in V$, we write $f(v) = v'$ for convenience) and the edge set E_1 of DG is defined as follows: The edge ab is in E if and only if both ab' and $a'b$ are edges in E_1 . Clearly the duplicate graph of the Twig graph is disconnected. In order to view as a connected graph we give the following definition from (Thirusangu *et al.*, 2010).

Definition EDG (T_m): Let $G(V,E)$ be a Twig graph T_m and let $DG = (V_1, E_1)$ where the vertex set $V_1 = V \cup V'$ and $V \cap V' = \phi$ be a Duplicate graph of a Twig T_m . Add an edge between any one vertex from V to any other vertex in V' , except the terminal vertices of V and V' . For convenience, let us take $v_2 \in V$ and $v'_2 \in V'$ and thus the edge (v_2, v'_2) is formed. We call this new graph as the Extended Duplicate Graph of the Twig T_m and it is denoted as $EDG(T_m)$. Clearly this $EDG(T_m)$ has $(6m+4)$ vertices and $(6m+3)$ edges.

Main results

In this section, we present an algorithm and prove the existence of (a,d)-edge-antimagic vertex labeling for $EDG(T_m)$.

Algorithm:

Input: EDG (T_m) with $(6m+4)$ vertices and $(6m+3)$ edges.

Step 1: Denote the $(6m+4)$ vertices as

$$V = \{v_1, v_2, \dots, v_{3m+1}, v_{3m+2}, v'_1, v'_2, \dots, v'_{3m+1}, v'_{3m+2}\}$$

Step 2: If 'm' is even, then $m = 2n; n \in \mathbb{N}$

If $n \in \mathbb{N}$, then the EDG (T_m) is of the form EDG (T_{2n}); $n \in \mathbb{N}$

Define $f: V \rightarrow \{1, 2, 3, \dots, 6m+4\}$ such that for $0 \leq i \leq \frac{m-2}{2}$

- (i) $f(v_{6i+3}) = 6i + 3, f(v_{6i+4}) = 6i + 5, f(v_{6i+5}) = 6i + 7$
 - (ii) $f(v_{6i+6}) = 6(m-i) + 1, f(v_{6i+7}) = 6(m-i) - 1, f(v_{6i+8}) = 6(m-i) - 3$
 - (iii) $f(v'_{6i+3}) = 6(m-i) + 2, f(v'_{6i+4}) = 6(m-i), f(v'_{6i+5}) = 6(m-i) - 2$
 - (iv) $f(v'_{6i+6}) = 6i + 4, f(v'_{6i+7}) = 6i + 6, f(v'_{6i+8}) = 6i + 8$
- and also
- $$f(v_1) = 1, f(v_2) = 6m + 3, f(v'_1) = 6m + 4, f(v'_2) = 2$$

Step 3:

If m is odd, then $m = 2n - 1; n \in \mathbb{N}$

The EDG (T_m) is of the form EDG (T_{2n-1}); $n \in \mathbb{N}$

Define $f: V \rightarrow \{1, 2, 3, \dots, 6m+4\}$ for the functions defined as

in Step 2 for (i) to (iv) with the limits $0 \leq i \leq \frac{m-1}{2}$,
 $0 \leq i \leq \frac{m-3}{2}$, $0 \leq i \leq \frac{m-1}{2}$ and $0 \leq i \leq \frac{m-3}{2}$ respectively.

Step 4:

Define $f^*: E \rightarrow \mathbb{N}$ such that

$$f^*(v_i, v_j) = |f(v_i) + f(v_j)|$$

Output: Labeled EDG (T_m).

Theorem:

The Extended Duplicate graph of Twig (T_m), $m \geq 1$ admits (a,d)-edge-antimagic vertex labeling.

Proof:

Let EDG (T_m) be a Extended duplicate graph of the Twig (T_m). Clearly EDG (T_m) has $(6m+4)$ vertices and $(6m+3)$ edges.

Denote the set of vertices as

$$V = \{v_1, v_2, \dots, v_{3m+1}, v_{3m+2}, v'_1, v'_2, \dots, v'_{3m+1}, v'_{3m+2}\}$$

Case (1)

Let T_m be a Twig where $m = 2n, n \in \mathbb{N}$.

Consider the Twig of the type $T_{2n}, n \in \mathbb{N}$.

In this case we get Twig graphs $T_2, T_4, T_6, T_8, \dots$

To label the vertices of V, define a map $f: V \rightarrow \{1, 2, 3, \dots, 6m+4\}$ as given in **Step 2** of the above algorithm.

The vertices v_1 and v_2 are labeled as 1 and $6m+3$; the vertices

$$(v_3, v_4, v_5), (v_9, v_{10}, v_{11}), \dots, (v_{3m-3}, v_{3m-2}, v_{3m-1}), (v_{3m+2}, v_{3m+1}, v_{3m}), (v_{3m-4}, v_{3m-5}, v_{3m-6}), \dots, (v_8, v_7, v_6)$$

receive consecutive odd numbers such as 3, 5, 7, ..., $6m-3, 6m-1, 6m+1$ as labels; the vertices v'_1 and v'_2 are labeled as $6m+4$ and 2; the vertices

$$(v'_3, v'_4, v'_5), (v'_9, v'_{10}, v'_{11}), \dots, (v'_{3m-3}, v'_{3m-2}, v'_{3m-1}), (v'_{3m+2}, v'_{3m+1}, v'_{3m}), (v'_{3m-4}, v'_{3m-5}, v'_{3m-6}), \dots, (v'_8, v'_7, v'_6)$$

receive consecutive even numbers such as $6m+2, 6m, 6m-2, \dots, 8, 6, 4$, as labels.

Thus all the $6m+4$ vertices are labeled.

From the definition of EDG (T_m), the $(6m+3)$ edges of EDG (T_m) are of the form

$$(v_1, v_2), (v_2, v_1), (v_2, v_2), (v_{2+3i}, v'_{3i+3+j}), (v'_{2+3i}, v_{3i+3+j})$$

for $0 \leq j \leq 2, 0 \leq i \leq m-1$.

The induced function $f^*: E \rightarrow \mathbb{N}$ is defined such that $f^*(v_i, v_j) = |f(v_i) + f(v_j)|$.

Thus the edges

$$(v_1, v_2), (v_2, v_3), (v_2, v_4), (v_2, v_5), (v_5, v_6), (v_5, v_7), (v_5, v_8), \dots, (v_{3m-1}, v_{3m}), (v_{3m-1}, v_{3m+1}), (v_{3m-1}, v'_{3m+2})$$

receive consecutive odd numbers such as 3, 5, 7, 9, ..., $6m-1, 6m+1, 6m+3$ as labels; the edge

(v_2, v'_2) receive the number $6m+5$ as label; the edges

$$(v_{3m+2}, v'_{3m-1}), (v_{3m+1}, v'_{3m-1}), (v_{3m}, v'_{3m-1}), (v'_{3m-1}, v_{3m-4}), (v'_{3m-2}, v_{3m-4}), (v'_{3m-3}, v_{3m-4}), \dots, (v'_5, v_2), (v'_4, v_2), (v'_3, v_2), (v_2, v'_1)$$

receive consecutive odd numbers such as $6m+7, 6m+9, 6m+11, \dots, 12m+1, 12m+3, 12m+5, 12m+7$ as labels.

Thus all the edges are labeled as $(3, 5, 7, \dots, 12m+7)$ and all are distinct.



Fig. 1. EDG (T_6) (a,d)-edge antimagic vertex.

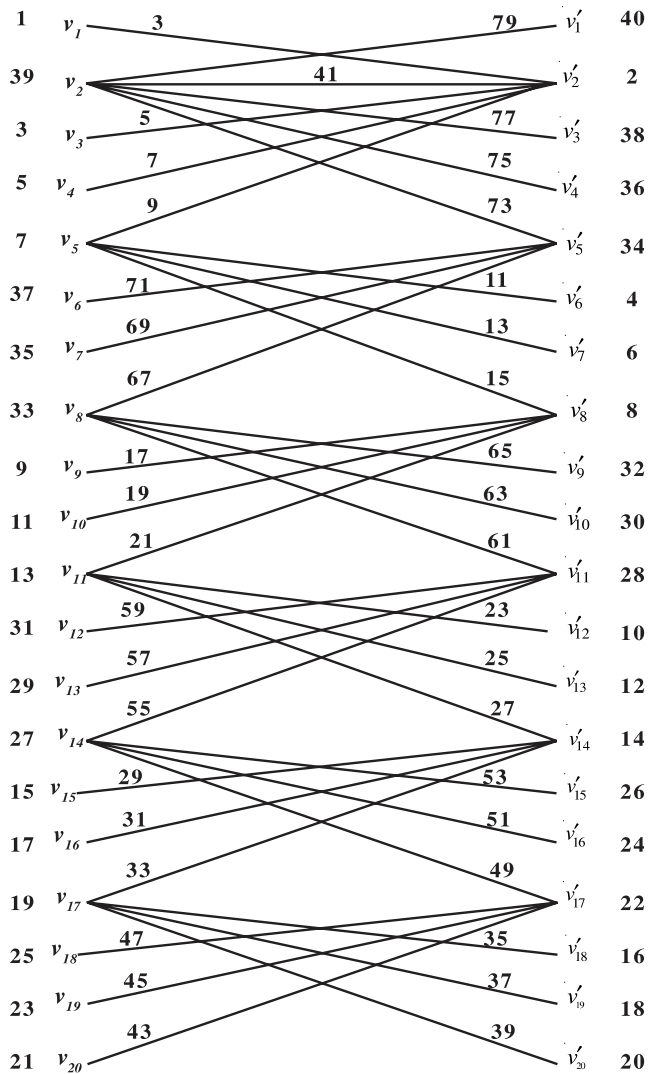
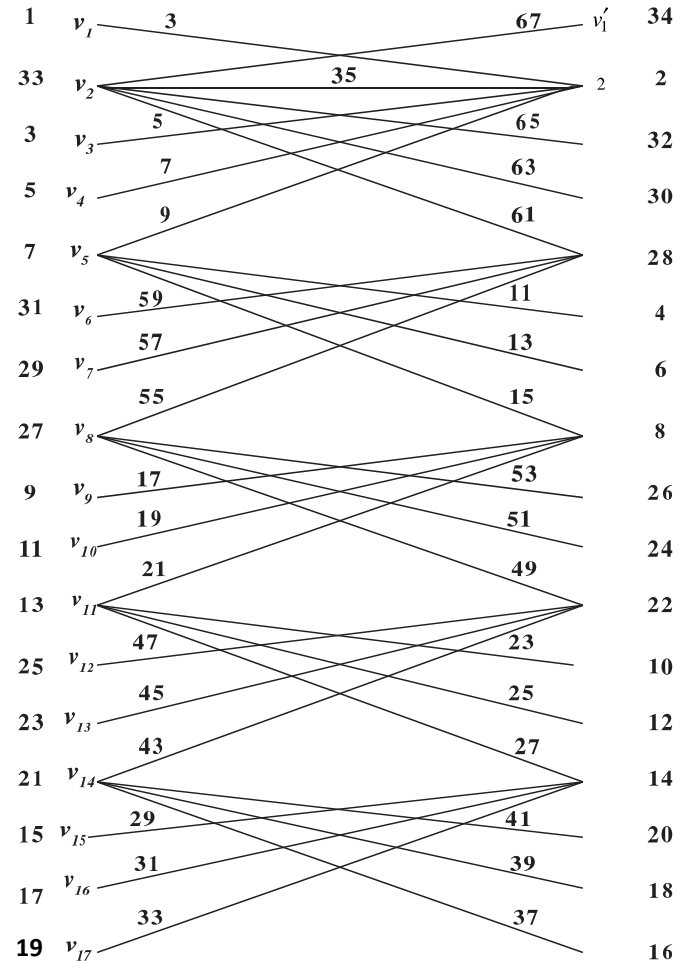


Fig. 2. EDG (T_5) (a,d)-edge antimagic vertex labeling.



Hence all the edges are labeled as $\{a, a+d, a+2d, \dots, a+(|E|-1)d\}$, where $a=3, d=2$ and all are distinct (The whole idea of the proof is illustrated in Fig. 1).

Hence $EDG(T_m)$, where $m=2n; n \in \mathbb{N}$ admits (a,d) -edge-antimagic vertex labeling.

Case 2:

Let T_m be a Twig, where $m=2n-1, n \in \mathbb{N}$.

Consider the Twig of the type $T_{2n-1}, n \in \mathbb{N}$.

In this case we get Twig graphs $T_1, T_3, T_5, T_7, \dots$

To label the vertices of V , define a map $f: V \rightarrow \{1, 2, 3, \dots, 6m+4\}$ as given in **Step 3** of the above algorithm. The vertices v_1 and v_2 are labeled as 1 and $6m+3$; the vertices

$(v_3, v_4, v_5), (v_9, v_{10}, v_{11}), \dots, (v_{3m}, v_{3m+1}, v_{3m+2}), (v_{3m-1}, v_{3m-2}, v_{3m-3}), \dots, (v_8, v_7, v_6)$ receive consecutive odd numbers such as 3, 5, 7, ..., $6m-3, 6m-1, 6m+1$ as labels; the vertices v'_1 and v'_2 are labeled as $6m+4$ and 2; the vertices

$(v'_3, v'_4, v'_5), (v'_9, v'_{10}, v'_{11}), \dots, (v'_{3m}, v'_{3m+1}, v'_{3m+2}), (v'_{3m-1}, v'_{3m-2}, v'_{3m-3}), \dots, (v'_8, v'_7, v'_6)$ receive consecutive even numbers such as $6m+2, 6m, 6m-2, \dots, 8, 6, 4$ as labels.

Thus all the $6m+4$ vertices are labeled.

From the definition of $EDG(T_m)$, the $(6m+3)$ edges of

$EDG(T_m)$ are of the form

$$(v_1, v'_2), (v_2, v'_1), (v_2, v'_2), (v_{2+3i}, v'_{3i+3+j}), (v_{2+3i}, v'_{3i+3+j})$$

for $0 \leq j \leq 2, 0 \leq i \leq m-1$.

The induced function $f^* : E \rightarrow \mathbb{N}$ is defined such that

$$f^*(v_i, v_j) = |f(v_i) + f(v_j)|$$

Thus the edges

$(v_1, v'_2), (v'_2, v_3), (v'_2, v_4), (v'_2, v_5), (v_5, v'_6), (v_5, v'_7), (v_5, v'_8), \dots, (v'_{3m-1}, v_{3m}), (v'_{3m-1}, v_{3m+1}), (v'_{3m-1}, v_{3m+2})$

receive consecutive odd numbers such as 3, 5, 7, 9, ..., $6m-1, 6m+1, 6m+3$ as labels; the edge

(v_2, v'_2) receive the number $6m+5$ as label; the edges

$(v'_{3m+2}, v_{3m-1}), (v'_{3m+1}, v_{3m-1}), (v'_{3m}, v_{3m-1}), (v'_{3m-1}, v_{3m-4}), (v'_{3m-2}, v_{3m-4}), (v'_{3m-3}, v_{3m-4}), \dots, (v'_5, v_2), (v'_4, v_2), (v'_3, v_2), (v_2, v'_1)$ receive consecutive odd

numbers such as $6m+7, 6m+9, 6m+11, \dots, 12m+1, 12m+3, 12m+5, 12m+7$ as labels.

Thus all the edges are labeled as $(3, 5, 7, \dots, 12m+7)$ and all are distinct.

Hence all the edges are labeled as

$\{a, a+d, a+2d, \dots, a+(|E|-1)d\}$, where $a=3, d=2$ and all are distinct (The above concept is illustrated in Fig. 2).

Hence $EDG(T_m)$, where $m=2n-1; n \in \mathbb{N}$ admits

(a,d) -edge-antimagic vertex labeling.

Hence the proof of the theorem is completed.

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