Indian Journal of Science and Technology Vol. 4 No. 2 (Feb 2011) ISSN: 0974- 6846

Graceful and Skolem graceful Labeling in Extended Duplicate Graph of Path

P. P. Ulaganathan, K. Thirusangu, and B. Selvam

Department of Mathematics, S.I.V.E.T. College, Gowrivakkam, Chennai-600 073, India
ppulaganathan@gmail.com, kthirusangu@gmail.com, chellamaths@yahoo.com

Abstract

In this paper, we prove that the Extended Duplicate Graph of Path is Graceful and Skolem-graceful.
AMS subject classification: 05C78

Keywords: Graph labeling, Graceful labeling, Skolem-graceful labeling, Extended Duplicate Graph.

Introduction

In 1967, the concept of graph labeling was introduced by Rosa (1997). Graph labeling is an assignment of integers to the edges or vertices or both subject to certain conditions. Labeled graphs serve as useful models in a broad range of applications such as circuit design, communication network addressing, X-ray crystallography, radar, astronomy, data base management and coding theory. Over the past three decades various labeling of graphs such as cordial labeling, prime labeling, binary labeling, magic labeling, anti-magic labeling, bi-magic labeling, mean labeling, arithmetic labeling, graceful labeling, harmonious labeling etc., have been investigated in the literature (Gallian, 2010). Golomb has introduced graceful labeling (Golomb, 1972). In 1989 Gallian showed that all mobius ladders are graceful. Aravamudhan and Murugan have shown that the complete tripartite graph $K_{1,m,n}$ is both graceful and harmonious. Lee has introduced Skolem-graceful labeling. Lee *et al.* (1988) have shown that a connected graph is skolem-graceful if and only if it is a graceful tree (Lee *et al.,* 1988). They also proved that the disjoint union of 2 or 3 stars is skolem-graceful if and only if atleast one star has even size. Youssef (2003) proved that if G is skolem-graceful then $G+\bar{K}_n$ is graceful (Youssef, 2003). In Youssef (2003) it is proved that extended duplicate graph of paths are cordial. Though graceful labeling and Skolem-graceful labeling have been studied for different kinds of graphs, the graceful and skolem-graceful labeling for duplicate graphs have not been investigated. In this paper we prove that extended duplicate graph of path graphs are graceful and Skolem graceful.

Preliminaries

In this section, we give the basic notions relevant to this paper. Let G = G (V, E) be a finite, simple and undirected graph with p vertices and q edges. By a labeling we mean one-to-one mapping that carries a set of graph elements onto a set of numbers, called labels (usually the set of integers). In this paper we deal labeling with domain as the set of all vertices.

Definition: A function f is said to be graceful of a graph G with q-edges if *f* is 1-1 from $V \rightarrow \{0, 1, 2, \ldots, q\}$ such that for each edge *xy* assigned the label $| f(x) - f(y) |$, the resulting edge labels are distinct numbers {1, 2, …, q}.

Definition: A (p, q) graph G is called Skolem-graceful if there exists 1-1 map $f: V \rightarrow \{1, 2, \ldots, p\}$ such that the edge labels induced by $| f(x) - f(y) |$ for each *xy* are 1, 2, …, q. This is also called node-graceful. A necessary condition for a graph to be Skolem-graceful is that $p \geq q+1$.

Definition: Let G (V, E) be a simple graph. A duplicate graph of G is DG = (V_1, E_1) where the vertex set $V_1 = V \cup V'$ and $V \cap V' = \phi$ and $f: V \rightarrow V'$ is bijective (for $v \in V$, we write f (v) = v′ for convenience) and the edge set E_1 of DG is defined as follows: The edge ab is in E if and only if both ab′ and a′b are edges in E_1. Clearly the duplicate graph of the path graph is disconnected. We give the following definition from (Thirusangu *et al.,* 2010).

Definition: Let DG = (V_1, E_1) be a duplicate graph of the path graph G (V, E). We add an edge between any one vertex from V to any other vertex in V′, except the terminal vertices of V and V′. For convenience, let us take $v_2 \in V$ and $v_2^1 \in V'$ and thus the edge (v_2, v_2^1) is formed. This graph is called the extended duplicate graph of the path P_m and it is denoted by EDG (P_m).

Main results

Graceful labeling for EDG (P_m): In this section, we present an algorithm and prove the existence of graceful labeling for EDG (P_m).

Algorithm:

Input: EDG (P_m) with p vertices and q edges where p = (2m+2), q = (2m+1).

Step 1: Denote the (2m+2) vertices as V = {v_1, v_2, …, v_m, v_{m+1}, $v'_1, v'_2 \ldots v'_m, v'_{m+1}$}.

Step 2: If ‘m’ is even, then m = 2n; $n \in N$. The EDG (P_m) is of the form

EDG (P_{2n}); $n \in N$.

Define $f: V \rightarrow \{0,1,2,\ldots,q\}$ such that

Indian Journal of Science and Technology Vol. 4 No. 2 (Feb 2011) ISSN: 0974- 6846

$f(v_{2i+1}) = i;\ \ 0 \le i \le m/2$

$f(v_{2i+2}) = m-i;\ 0 \le i \le (m-2)/2$

$f(v'_{2i+1}) = (m+1)+i;\ 0 \le i \le m/2$

$f(v'_{2i+2}) = (2m+1)-i;\ \ 0 \le i \le (m-2)/2$

Step 3: If 'm' is odd, then m = 2n+1; $n \in N$.

The EDG (P_m) is of the form EDG (P_{2n+1}); $n \in N$.

Define $f: V \to \{0, 1, 2,\ldots, q\}$ such that

$f(v_{2i+1}) = i;\ \ 0 \le i \le (m-1)/2$

$f(v_{2i+2}) = m-i;\ \ 0 \le i \le (m-1)/2$

$f(v'_{2i+1}) = (m+1)+i;\ 0 \le i \le (m-1)/2$

$f(v'_{2i+2}) = (2m+1)-i;\ 0 \le i \le (m-1)/2$

Step 4: Define $f^*: E \to N$ such that $f^*(v_i, v_j) =$

$| f(v_i) - f(v_j) |$

Output: Graceful labeling of EDG (P_m)

Theorem: The extended duplicate graph of path P_m, $m \ge 2$ admits graceful labeling.

Proof: Let EDG (P_m) be a Extended duplicate graph of path P_m. Clearly EDG (P_m) has p vertices and q edges where
p = (2m+2), q = (2m+1). Denote the set of vertices as

$V = \{v_1, v_2, \ldots, v_m, v_{m+1}, v'_1, v'_2 \ldots v'_m, v'_{m+1}\}$

Case 1: In this case, we prove this theorem for even paths. Let P_m be a path, where m=2n; $n \in N$.

Consider the paths of the type P_{2n}; $n \in N$. In this case, we get path graphs P_2, P_4, P_6, …

To get graceful labeling, define a map $f: V \to \{0,1, 2, \ldots, q\}$ as given in step 2 of the above algorithm.

Therefore the vertices $v_1, v_3, v_5, \ldots, v_{m+1}, v_m, v_{m-2}, \ldots, v_2$ receive consecutive numbers such as 0, 1, 2, …, m as labels and $v'_1, v'_3, v'_5 \ldots v'_{m+1}, v'_m, v'_{m-2}, \ldots, v'_2$ receive consecutive numbers such as m+1, m+2, …, 2m+1.

Thus all the (2m+2) vertices are labeled.

From the definition of EDG (P_m), the (2m+1) edges of EDG (P_m) are of the form (v_i, v'_{i+1}) (v'_i, v_{i+1}) for i=1, 2, …, m and (v_2, v'_2). Define induced function $f^*: E \to N$ such that $f^*(v_i, v_j) = | f(v_i) - f(v_j) |$

Now the edges are labeled as follows: The edges are in the following five forms, namely, (v_{2i+1}, v'_{2i+2}), (v'_{2i+1}, v_{2i+2}) where $0 \le i \le (m-2)/2$; (v_{2i+2}, v'_{2j+1}), (v'_{2i+2}, v_{2j+1}) where j=i+1, $0 \le i \le (m-2)/2$ and (v_2, v'_2).

For any i, where $0 \le i \le (m-2)/2$,

$f^*(v_{2i+1}, v'_{2i+2}) = | f(v_{2i+1}) - f(v'_{2i+2}) | = | 2i - (2m+1 - i) |$

$= | q - 2i |$ which is an odd number $\forall$ i

These m/2 edges are labeled with numbers q, q-2, q-4, …, (q+5)/2.

For any i, where $0 \le i \le (m-2)/2$,

$f^*(v'_{2i+1}, v_{2i+2}) = | \{(m+1)+i\} - (m-i) |$

$= |2i + 1|$ which is an odd number $\forall$ i

These m/2 edges are labeled with numbers 1, 3, 5, …, (q-3)/2.

Let j = i+1, for any i, where $0 \le i \le (m-2)/2$,

$f^*(v_{2i+2}, v'_{2j+1}) = | f(v_{2i+2}) - f(v'_{2j+1}) | =$

$| (m - i) - (m+1+j) |$

$= | 2i + 2 |$ which is an even number $\forall$ i

These m/2 edges are labeled with numbers 2, 4, 6, …, (q-1)/2.

Let j = i+1, for any i, where $0 \le i \le (m-2)/2$,

$f^*(v'_{2i+2}, v_{2j+1}) = | f(v'_{2i+2}) - f(v_{2j+1}) | = | (2m+1 - i) - j |$

$= | (q-1) - 2i |$ which is an even number $\forall$ i

These m/2 edges are labeled with numbers q-1, q-3, q-5, …, (q+3)/2.

For the edge (v_2, v'_2),

$f^*(v_2, v'_2) = | f(v_2) - f(v'_2) | = | m - (2m + 1) |$

$= (q+1)/2$ which is also an odd number.

Thus the edges (v'_1, v_2), (v_2, v'_3), (v'_3, v_4) ,…, (v_m, v'_{m+1}) receive consecutive numbers such as 1, 2, …, (q-1)/2 as labels; the edge (v_2, v'_2) receives the number (q+1)/2 as label and (v_{m+1}, v'_m), (v'_m, v_{m-1}) ,…, (v'_2, v_1) receive consecutive numbers such as (q+3)/2, (q+5)/2, …, q as labels.

That is $f^*(E) = \{1, 2, 3, \ldots, q\}$ which are all distinct and hence EDG (P_m) is graceful.

Case (2): In this case, we prove this theorem for odd paths. Let P_m be a path, where m = 2n+1; $n \in N$.

Consider the paths of the type P_{2n+1}; $n \in N$. In this case, we get path graphs P_3, P_5, P_7, ….

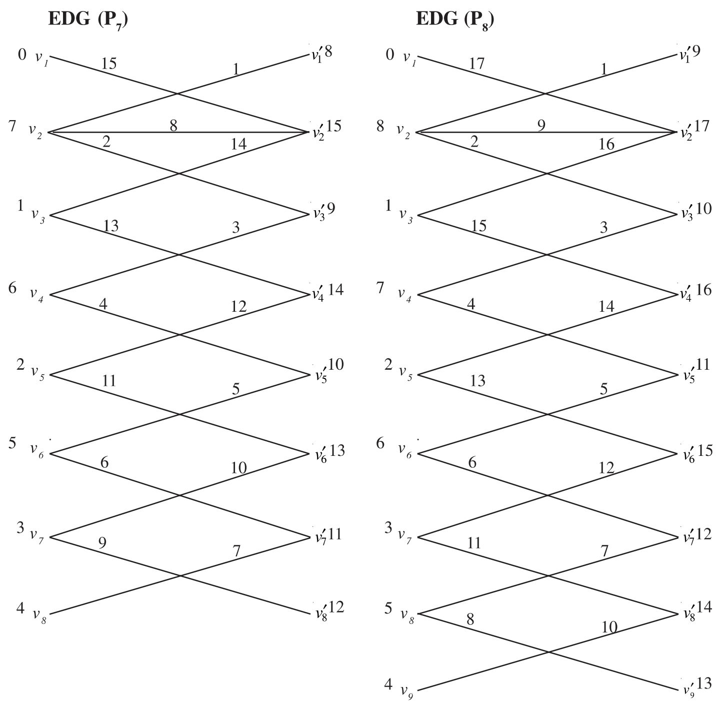
To get graceful labeling, define a map $f: V \to \{0, 1, 2, \ldots, q\}$ as given in step 3 of the above algorithm.

Therefore the vertices $v_1, v_3, v_5, \ldots v_m, v_{m+1}, v_{m-1}, v_{m-3}, \ldots, v_2$ receive consecutive numbers such as 0, 1, 2, … m as

Indian Journal of Science and Technology Vol. 4 No. 2 (Feb 2011) ISSN: 0974- 6846

labels and $v'_1, v'_3, \ldots v'_m, v'_{m+1}, v'_{m-1}, v'_{m-3}, \ldots, v'_2$ receive consecutive numbers such as m+1, m+2, … , 2m+1 as

Annexure I. Graceful labeling.



labels.

Thus all the (2m+2) vertices are labeled.

Define induced function $f^*: E \rightarrow N$ such that $f^*(v_i, v_j) = | f(v_i) - f(v_j) |$

Now the edges are labeled as follows: The edges are in the following five forms, namely, (v_{2i+1}, v'_{2i+2}),

(v'_{2i+1}, v_{2i+2}) where $0 \leq i \leq (m-1)/2$; (v_{2i+2}, v'_{2j+1}),

(v'_{2i+2}, v_{2j+1}) where j=i+1, $0 \leq i \leq (m-3)/2$ and (v_2, v'_2).

For any i, where $0 \leq i \leq (m-1)/2$,

$f^*(v_{2i+1}, v'_{2i+2}) = | f(v_{2i+1}) - f(v'_{2i+2}) | = | i - (2m+1 - i) |$

$= | (q - 2i) |$ which is an odd number $\forall$ i.

These (m+1)/2 edges are labeled with numbers as q, q-2, q-4, …, (q+3)/2.

For any i, where $0 \leq i \leq (m-1)/2$,

$f^*(v'_{2i+1}, v_{2i+2}) = | f(v'_{2i+1}) - f(v_{2i+2}) | = | (m + 1 + i) - (m - i)|$

$= | 2i + 1 |$ which is an odd number $\forall$ i

These (m+1)/2 edges are labeled with numbers as 1,3,5,…,(q-1)/2.

Let j = i+1, for any i, where $0 \leq i \leq (m-3)/2$,

$f^*(v_{2i+2}, v'_{2j+1}) = | f(v_{2i+2}) - f(v'_{2j+1}) | = | (m - i) - (m+1+j) |$

$= | 2i + 2 |$ which is an odd number $\forall$ i

These (m-1)/2 edges are labeled with numbers as 2, 4, 6, …, (q-3)/2.

For any i, where $0 \leq i \leq m-3/2$,

$f^*(v'_{2i+2}, v_{2j+1}) = | f(v'_{2i+2}) - f(v_{2j+1}) | = | (2m+1-i) - j |$

$= |(q-1) - 2i |$ which is an even number $\forall$ i

These (m-1)/2 edges are labeled with numbers as q-1, q-3, q-5, …, (q+5)/2.

For the edge (v_2, v'_2),

$f^*(v_2, v'_2) = | f(v_2) - f(v'_2) | = | m - (2m + 1) |$

$= (q+1)/2$ which is also an even number.

Thus the edges (v'_1, v_2), (v_2, v'_3) (v'_3, v_4) … (v'_m, v_{m+1}) receive consecutive numbers such as 1, 2, …, (q-1)/2 as labels; the edge (v_2, v'_2) receives the number (q+1)/2 as label and the edges (v'_{m+1}, v_m), (v_m, v'_{m-1}), (v'_{m-1}, v_{m-2}), … ,(v'_2, v_1) receive consecutive numbers such as (q+3)/2, (q+5)/2, …, q as labels.

That is $f^*(E) = \{1, 2, 3, …, q\}$ which are all distinct and hence EDG (P_m) is graceful.

Illustration: Graceful labeling for the graphs EDG (P_7) and EDG (P_8) are shown in the annexure I.

Skolem-graceful labeling for EDG (P_m)

In this section, we present an algorithm and prove the existence of Skolem-Graceful labeling for EDG (P_m).

Algorithm:

Input: EDG (P_m) with p vertices and q edges where p = (2m + 2), q = (2m + 1).

Step 1: Denote the (2m+2) vertices as $V = \{v_1, v_2, …, v_m, v_{m+1}, v'_1, v'_2 \ldots v'_m, v'_{m+1}\}$

Step 2: If 'm' is even, then m = 2n; $n \in N$. The EDG (P_m) is of the form

EDG (P_{2n}); $n \in N$.

Define $f: V \rightarrow \{1,2,…,p\}$

$f(v_{2i-1}) = i$; $1 \leq i \leq (m+2)/2$

$f(v_{2i}) = (m+2)-i$; $1 \leq i \leq m/2$

$f(v'_{2i-1}) = (m+1)+i$; $1 \leq i \leq (m+2)/2$

$f(v'_{2i}) = (2m+3)-i$; $1 \leq i \leq m/2$

Step 3: If 'm' is odd, then m = 2n+1; $n \in N$.

The EDG (P_m) is of the form EDG (P_{2n+1}); $n \in N$.

Define f: V → {1,2,…,p}

$f(v_{2i-1}) = i;\ 1 \le i \le (m+1)/2$

$f(v_{2i}) = (m+2)-i;\ 1 \le i \le (m+1)/2$

$f(v'_{2i-1}) = (m+1)+i;\ 1 \le i \le (m+1)/2$

$f(v'_{2i}) = (2m+3)-i;\ 1 \le i \le (m+1)/2$

Step 4: Define $f^*: E \to N$ such that $f^*(v_i, v_j) = | f(v_i) - f(v_j) |$

Output: Skolem graceful labeling of EDG (P_m)

Theorem: The extended duplicate graph of path P_m, $m \ge 2$ admits skolem – graceful labeling.

Proof: Let EDG (P_m) be a Extended duplicate graph of path P_m. Clearly EDG (P_m) has p vertices and q edges where p = (2m+2), q = (2m+1).

Denote the set of vertices as V = ($v_1, v_2, \ldots, v_m, v_{m+1}, v'_1, v'_2 \ldots v'_m, v'_{m+1}$}

Case 1: In this case, we prove this theorem for even paths. Let P_m be a path, when m=2n; $n \in N$.

Consider the paths of the type P_{2n}; $n \in N$. In this case, we get path graphs $P_2, P_4, P_6, \ldots$

To get the Skolem graceful labeling, define a map f: V → {1, 2, …, p} as given in step 2 of the above algorithm.

Therefore the vertices $v_1, v_3, v_5, \ldots, v_{m+1}, v_m, v_{m-2}, \ldots, v_2$ receive consecutive numbers such as 1, 2, …, (m+1) as labels and $v'_1, v'_3, v'_5 \ldots v'_{m+1}, v'_m, v'_{m-2}, \ldots, v'_2$ receive consecutive numbers such as m+2, m+3, …, 2m+2.

Thus all the (2m+2) vertices are labeled as 1, 2, 3, …, p.

From the definition of EDG (P_m), the (2m+1) edges of EDG (P_m) are of the form (v_i, v'_{i+1}), (v'_i, v_{i+1}) for i=1, 2, …, m and (v_2, v'_2).

Define the induced function $f^*: E \to N$ such that $f^*(v_i, v_j) = | f(v_i) - f(v_j) |$

Now the edges are labeled as follows : The edges are in the following five forms, namely, (v_{2i-1}, v'_{2i}), (v'_{2i-1}, v_{2i}) where $1 \le i \le m/2$; (v_{2i}, v'_{2j-1}), (v'_{2i}, v_{2j-1}) where j=i+1, $1 \le i \le m/2$ and (v_2, v'_2).

For any i, where $1 \le i \le m/2$,

$f^*(v_{2i-1}, v'_{2i}) = | f(v_{2i-1}) - f(v'_{2i}) | = | i - (2m+3 - i) |$

$= | (q+2) - 2i |$ which is an odd number ∀ i

These m/2 edges are labeled with numbers q, q-2, q-4, …, (q+5)/2.

For any i, where $1 \le i \le m/2$,

$f^*(v'_{2i-1}, v_{2i}) = | f(v'_{2i-1}) - f(v_{2i}) | =$
$| (m + 1 + i) - (m+2-i) | = |2i - 1|$ which is an odd number ∀ i

These m/2 edges are labeled with numbers 1, 3, 5, …, (q-3)/2.

Let j = i+1, for any i, where $1 \le i \le m/2$,

$f^*(v_{2i}, v'_{2j-1}) = | f(v_{2i}) - f(v'_{2j-1}) | =$
$| (m+2-i) - (m+1+j) | = | 2i |$ which is an even number ∀ i

These m/2 edges are labeled with numbers 2, 4, 6, …, (q-1)/2.

Let j = i+1, for any i, where $1 \le i \le m/2$,

$f^*(v'_{2i}, v_{2j-1}) = | f(v'_{2i}) - f(v_{2j-1}) | = | (2m+3 - i) - j |$

$= | (q+1) - 2i |$ which is an even number ∀ i

These m/2 edges are labeled with numbers | (q+1) – 2i | as q-1, q-3, q-5, …, (q+3)/2.

For the edge (v_2, v'_2),

$f^*(v_2, v'_2) = | f(v_2) - f(v'_2) | = | (m+2) - (2m + 3) |$

$= (q+1)/2$ which is also an odd number.

Thus the edges (v'_1, v_2), (v_2, v'_3), (v'_3, v_4) ,…, (v_m, v'_{m+1}) receive consecutive numbers such as 1, 2, 3, …, (q-1)/2 as label; the edge (v_2, v'_2) receives the number (q+1)/2 as label and the edges (v_{m+1}, v'_m), (v'_m, v_{m-1}) ,…, (v'_2, v_1) receive consecutive numbers such as (q+3)/2, (q+5)/2, …, q as labels.

That is f^* (E) = {1, 2, 3, …, q} which are all distinct and hence EDG (P_m) is Skolem graceful.

Case (2): In this case, we prove this theorem for odd paths. Let P_m be a path, where m = 2n+1; $n \in N$.

Consider the paths of the type P_{2n+1}; $n \in N$. In this case, we get path graphs $P_3, P_5, P_7, \ldots$.

To get the Skolem graceful labeling, define a map f: V → {1, 2, 3, …., p} as given in step 3 of the above algorithm.

Therefore the vertices $v_1, v_3, v_5, \ldots v_m, v_{m+1}, v_{m-3}, \ldots, v_2$ receive consecutive numbers such as 1, 2, 3, …, m+1 as labels and the vertices $v'_1, v'_3, \ldots v'_m, v'_{m+1}, v'_{m-1}, v'_{m-3}, \ldots, v'_2$ receive consecutive numbers such as m+2, m+3, … , (2m+2) as labels.

Thus all the (2m+2) vertices are labeled as 1, 2, 3, …, p.

Define induced function $f^*: E \to N$ such that $f^*(v_i, v_j) = | f(v_i) - f(v_j) |$

Now the edges are labeled as follows: The edges are in the following five forms, namely, (v_{2i-1}, v'_{2i}), (v'_{2i-1}, v_{2i}) where $1 \le i \le (m+1)/2$; (v_{2i}, v'_{2j-1}), (v'_{2i}, v_{2j-1}) where j=i+1, $1 \le i \le (m+1)/2$ and (v_2, v'_2).

For any i, where $1 \le i \le (m+1)/2$,
$f^*(v_{2i-1}, v'_{2i}) = | f(v_{2i-1}) - f(v'_{2i}) | = | i - (2m+3 - i) |$
$= | (q+2) - 2i |$ which is an odd number $\forall$ i
These (m+1)/2 edges are labeled with numbers q, q-2, q-4, …, (q+3)/2.

For any i, where $1 \le i \le (m+1)/2$,
$f^*(v'_{2i-1}, v_{2i}) = | f(v'_{2i-1}) - f(v_{2i}) | =$
$| (m + 1 + i) - (m+2-i) | = | 2i - 1 |$ which is an odd number $\forall$ i

These (m+1)/2 edges are labeled with numbers 1, 3, 5, …, (q-1)/2.
Let j = i+1, for any i, where $1 \le i \le (m+1)/2$,
$f^*(v_{2i}, v'_{2j-1}) = | f(v_{2i}) - f(v'_{2j-1}) |$
$= | (m+2-i) - (m+1+j) |$

$= | 2i |$ which is an even number $\forall$ i
These (m-1)/2 edges are labeled with numbers 2, 4, 6, …, (q-3)/2.

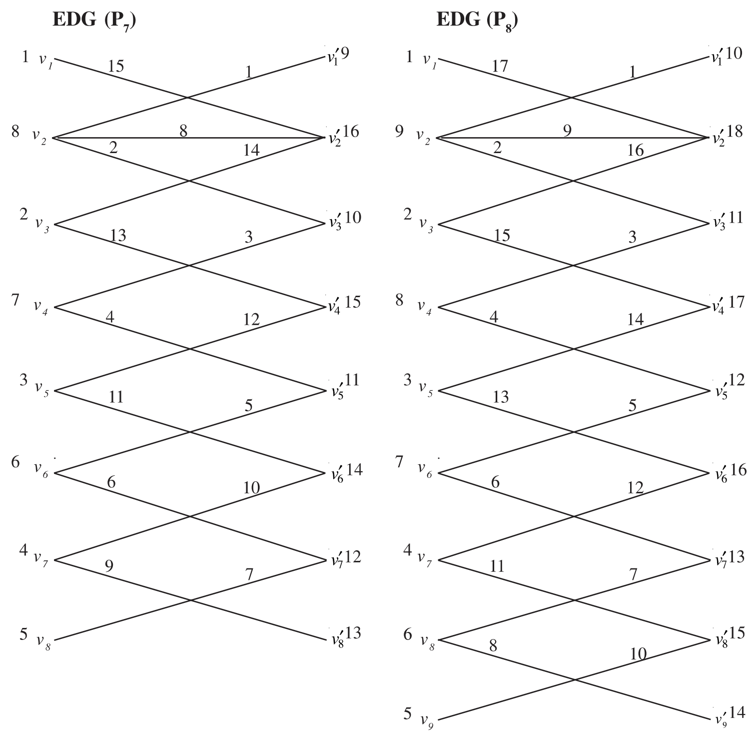
Let j = i+1, for any i, where $1 \le i \le (m+1)/2$,
$f^*(v'_{2i}, v_{2j-1}) = | f(v'_{2i}) - f(v_{2j-1}) | = | (2m+3 - i) - j |$
$= | (q+1) - 2i |$ which is an even number $\forall$ i
These (m-1)/2 edges are labeled with numbers q-1, q-3, q-5, …, (q+5)/2 which are distinct even number.
For the edge (v_2, v'_2),
$f^*(v_2, v'_2) = | f(v_2) - f(v'_2) | = | (m+2) - (2m + 3) |$
$= (q+1)/2$ which is also an even number.

Thus the edges (v'_1, v_2), (v_2, v'_3), (v'_3, v_4) ,…, (v'_{m+1}, v_{m+1}) receive consecutive numbers such as 1, 2, 3, …, (q-1)/2 as labels; the edge (v_2, v'_2) receives the number (q+1)/2 as label and the edges (v'_{m+1}, v_m), (v_m, v'_{m-1}), (v'_{m-1}, v_{m-2}) ,…, (v'_2 , v_1) receive consecutive numbers such as (q+3)/2, (q+5)/2, …, q as labels.

That is $f^*(E) = \{1, 2, 3, …, q\}$ which are all distinct and hence EDG (P_m) is Skolem graceful.

Illustration: Skolem-graceful labeling for the graph EDG (P_7) and EDG (P_8) are shown in the annexure II.

Annexure II. Skolem graceful labeling.



References

1. Gallian JA (2010) A dynamic survey of graph labeling. *Elec. J. Combin.* 17, # DS6.
2. Golomb SW (1997) How to number a graph, in graph theory and computing, RC Read, ed., Academic press, NY. pp:23-37.
3. Lee SM, Wang S and Wui I (1988) On Skolem-gracefulness of 4 stars. *Congr. Number.* 62, 235-239.
4. Rosa A (1967) On certain valuations of the vertices of a graph, theory of graphs (Int. Symp. Rome, July 1966), Gordon and Breech NY and Dunod paris. pp:349-355.
5. Thirusangu K, Ulaganathan PP and Selvam B (2010) Cordial labeling in duplicate graphs. *Int. J. Computer Math. Sci. Appl.* 4(1-2), 179-186.
6. Youssef MZ (2003) New families of graceful graphs. *Ars. Combin.* 67, 303-311.