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Approximation by a kind of B-spline method

Mehdi Zamani
Civil Engineering Department, Faculty of Technical and Engineering Moallem Square, Daneshjoo Avenue, Yasouj University, Yasouj, Kohgilooyeh and Boyerahmad-75918-74831, Iran
mahdi@mail.yu.ac.ir

Abstract

We present a model which creates a linear system of equations of symmetric and heptadiagonal. Therefore, for huge volume of data, the governing matrix is sparse and has a lot of zero entries, hence there is no need to save the entire values of matrix A. It is necessary to keep the bandwidth of data with the thickness of 4n. The non-zero entries part for a linear system with dimensions 100*100 is about 4%. For testing the presented model, 4 problems with highly non-linearity of data variations are selected. The results show the applicability, the efficiency and the simplicity of the model for approximation of highly non-linear data.

Keywords: Approximation, B-spline, heptadiagonal matrix, bandwidth, non-linear.

Introduction

In engineering activities there are data which generally requires some equations for their approximation. These data usually have non-uniform and non-linear distributions or variations. The classical methods of approximation often cannot predict the real behaviour of such data. The advanced methods of approximation such as Bezier, B-spline, Dierckx and de Boor methods are more applicable to the computer graphic approach. These methods don't have a suitable and sufficient efficiency for the approximation of laboratory and field engineering data. In this research an approximation method with using B-spline approach considered where it is comparable respect to the simplicity and operation calculations volume regarding to the above methods.

Although the literature about the methods of approximation is well documented (de Boor, 1978; Buren & Fairs, 1989; Lobo, 1995; Fomel, 1997), the method is presented by Dierckx (1993) may be more important. In his approach, B-spline function consists of basic polynomials with degrees 2, 3 and 4 or with orders 3, 4 and 5 respectively. The one with degree 3 cubic B-spline has the most application for approximation and estimation of data. The B-spline function for different orders is as follows:

$$s(t)=\sum_{k=0}^{l} c_k \, N_{k,m}(t), \qquad (1)$$

Where $N_{k,m}$ is k+1[th] basic function with order m, $l+1$ is the number of data pairs (points), and c_k's are the coefficients. The basic function can be obtained from the following recursive function,

$$N_{k,m}(t)=\frac{t-t_k}{t_{k+m-1}-t_k}N_{k,m-1}(t)+\frac{t_{k+m}-t}{t_{k+m}-t_{k+1}}N_{k+1,m-1}(t) \qquad (2)$$

The above recursive function for initial case where $m=1$ and $k=0,1,2,\cdots,l$ is,

$$N_{k+1}(t)=\begin{cases}1 & , \quad t_k \prec t \prec t_{k+1}\\ 0 & , \quad otherwise\end{cases} \qquad (3)$$

In this approach the number of control points is $l+m+1$ where they start from t_0 and end to t_{l+m+1}. The control points consist of three parts. The first part consists of one's which are from t_0 to t_{m-1} and have zero values. The second part is from t_m to t_l where $t_m=0$, $t_{m+1}=1$, $t_{m+2}=2$,...., $t_l=l-m+1$. The third series of control points consists of t_{l+1} to t_{l+m} with the values of $l-m+1$. For example these control points for 8 data points $l+1=8$ and the basic functions with degree 3 or order $m=4$ are the component of series $A=\{0,0,0;0,1,2,\ldots 3,4,5;5,5,5\}$. B-spline curves usually have more difficulty and complexity with respect to the general cubic spline curves. From the other approaches for determining these curves it is pointed to two cases of degrees 2 and 3 as follows (Salamon, 2006; Zamani, 2009a). Suppose there are n+1 data points (n+1 control points $p_0, p_1, p_2, \ldots, p_n$) and it is required to approximate them by B-spline curves of second degree. For doing this it is necessary to obtain $n-2$ B-spline curves with Eqs (4).

$$p_i(t)=(a_0+a_1t+a_2t^2)\,p_{i-1}+(b_0+b_1t+b_2t^2)\,p_i+(c_0+c_1t+c_2t^2)\,p_{i+1}, \qquad (4)$$

Where $i=1,2,3,\ldots,n-2$. The above quadratic B-spline curves have the continuity of C^1 at the boundaries segments. Eq. (4) in matrix notation is,

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$$p_i(t) = \lfloor t^2 \quad t \quad 1 \rfloor \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{bmatrix} \begin{Bmatrix} p_{i-1} \\ p_i \\ p_i \end{Bmatrix} = \lfloor t \rfloor [M] \{p\} \quad , t \in [0,1]$$

, (5)

The components of matrix M are obtained from the following continuity conditions for B-spline segments 1 and 2.

$$\begin{cases} p_1(t) = (a_0 + a_1 t + a_2 t^2) p_0 + (b_0 + b_1 t + b_2 t^2) p_1 + (c_0 + c_1 t + c_2 t^2) p_2 \\ p_2(t) = (a_0 + a_1 t + a_2 t^2) p_1 + (b_0 + b_1 t + b_2 t^2) p_2 + (c_0 + c_1 t + c_2 t^2) p_3 \end{cases} \quad (6)$$

$$\begin{cases} p_1(t) = p_0(t) \\ p_1^{'}(t) = p_2^{'}(t) \\ a_0 + a_1 t + a_2 t^2 + b_0 + b_1 t + b_2 t^2 + c_0 + c_1 t + c_2 t^2 = 1 \end{cases} \quad (7)$$

With including the continuity conditions Eq. (7) into Eq. (6) the matrix M is obtained as,

$$M = \frac{1}{2} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad (8)$$

The B-spline function Eq. (5) for segment i and $x \in [\lambda_i , \lambda_{i+1}]$ is,

$$p_i(x) = \lfloor x^2 \quad x \quad 1 \rfloor \begin{bmatrix} a^2 & 0 & 0 \\ 2ab & 0 & 0 \\ b^2 & b & 1 \end{bmatrix} \begin{bmatrix} 0.5 & -1 & 0.5 \\ -1 & 1 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix} \begin{Bmatrix} p_{i-1} \\ p_i \\ p_{i+1} \end{Bmatrix} = \lfloor x \rfloor [B][M] \{p\} \quad (9)$$

Where

$$b = \frac{\lambda_i}{\lambda_{i+1} - \lambda_i}, \quad a = \frac{1}{\lambda_{i+1} - \lambda_i} \quad and \quad t = ax + b$$

By using the same way as the above approach cubic B-spline curves function can be developed as,

$$p_i(t) = \lfloor x^3 \quad x^2 \quad x \quad 1 \rfloor \begin{bmatrix} a^3 & 0 & 0 & 0 \\ 3a^2 b & a^2 & 0 & 0 \\ 3ab^2 & 2ab & 0 & 0 \\ b^3 & b^2 & b & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{Bmatrix} p_{i-1} \\ p_i \\ p_{i+1} \\ p_{i+2} \end{Bmatrix} \quad (10)$$

Formulation

B-spline curves don't pass through the control points but move from near points. If the polygon that passes through the control point crosses itself, the B-spline curve crosses itself. By moving the positions of middle control points it is possible to change or correct the shape of B-spline curves. Therefore; it is possible to adjust the shape of B-spline curves. This has many applications for designing the curves and surfaces. When the data are from the engineering activities such as laboratory or field tests, they usually form a huge data. The mentioned methods of B-spline don't have the required efficiency for approximation of such data. The governing B-spline curves are not smooth and look like the forms of sinusoidal and vague curves. To overcome this problem it is highly recommended to apply the following formulation for approximation of data.

Consider the B- spline equation as summation of n+1 B-spline functions Eq. (11).

$$. s(x) = \sum_{i=0}^{n} c_i \, \beta_i [u_i(x)] = c_0 \beta_0(u_0) + c_1 \beta_1(u_1) + \ldots c_n \beta_n(u_n) \quad (11)$$

Where

$u \in [-2, 2]$, $u_i(x) = \dfrac{x - x_i}{h}$ and $h = \dfrac{b-a}{n}$ Each cubic B-spline function $\beta_i(u_i)$ is the summation of four cubic spline segments that have continuity C^2 along the boundaries curves (Saxena & Sahay, 2009) Eq. (12).

$$\beta_i(u) = \begin{cases} (2+u)^3 \quad , \quad -2 \le u \le -1 \\ 1 + 3(1+u) + 3(1+u)^2 - 3(1+u)^3 , \; -1 \le u \le 0 \\ 1 + 3(1-u) + 3(1-u)^2 - 3(1-u)^3 , \; 0 \le u \le 1 \\ (2-u)^3 \quad , \quad 1 \le u \le 2 \end{cases} \quad (12)$$

For increasing the continuity conditions of B-spline equation s(x) also its applicability the following exponential equation is chosen for segment curve (Zamani, 2009 b) (Fig. 1).

$$\beta_i(u) = a \, e^{-b u_i^2} - c , \quad (13)$$

Where

$a = 4.016478$, $b = 1.3740615$ and $c = 0.016478$.

In this model by the method of least square fitting approach and using Eq. (11), it can be written,

$$SSE = \sum_{i=1}^{m} [y_i - s(x_i)]^2 , \quad (14)$$

Where SSE is sum of square errors and m is the number of data pairs, $(x_i, y_i) \in \{(x_1, y_1), (x_2, y_2), \cdots, (x_m, y_m)\}$. Eq. (14) after substitution for $s(x_i)$ becomes,

$$SSE = \sum_{i=1}^{m} \left[y_i - \sum_{k=0}^{n} c_k \, \beta_k \right]^2 , \quad (15)$$

Where n+1 is the number of control points. Squaring Eq. (15) results,

$$SSE = \sum_{i=1}^{m} y_i^2 + \sum_{i=1}^{m} \sum_{j=0}^{n} \sum_{k=0}^{n} c_j \, c_k \, \beta_j^i \, \beta_k^i - 2 \sum_{i=1}^{m} \sum_{k=0}^{n} c_k \, \beta_k^i \, y_i \quad (16)$$

Taking the derivative of Eq. (16) respect to the coefficients c_k it results,

$$\frac{\partial SSE}{\partial c_j} = 2\sum_{i=1}^{m}\sum_{k=0}^{n} c_k\, \beta_k^i\, \beta_j^i - 2\sum_{i=1}^{m} \beta_j^i\, y_i = 0, \quad j = 0,1,2,\ldots,n \tag{17}$$

Eq. (17) forms a symmetric linear system of equations as,

$$\begin{bmatrix} \sum_{i=1}^{m}\beta_0^2 & \sum_{i=1}^{m}\beta_0\beta_1 & \cdots & \sum_{i=1}^{m}\beta_0\beta_n \\ \sum_{i=1}^{m}\beta_0\beta_1 & \sum_{i=1}^{m}\beta_1^2 & \cdots & \sum_{i=1}^{m}\beta_1\beta_n \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^{m}\beta_0\beta_n & \sum_{i=1}^{m}\beta_1\beta_n & \cdots & \sum_{i=1}^{m}\beta_n^2 \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{m}\beta_0 y_i \\ \sum_{i=1}^{m}\beta_1 y_i \\ \vdots \\ \sum_{i=1}^{m}\beta_n y_i \end{Bmatrix} = [A]\{c\} = \{b\} \tag{18}$$

With regarding to the basic function characteristics Eq. (12),

$$\sum_{i=1}^{m} \beta_i(x)\,\beta_j(x) = 0 \quad for \quad j \geq i+4 \ . \tag{19}$$

Therefore the linear system of equations (18) transforms to a heptadiagonal system of equations. This causes lots of zeroes for most of the components of matrix A (Fig. 2). There is no need to save all the components of matrix A regarding to its sparseness. A bandwidth of 4 or 4n components of matrix A are necessary for saving and following the calculations. This causes a lot of memory saving for keeping matrix A. For testing, checking and verifying the presented model about four problems are selected and shown here.

Problem 1.

The problem consists of 11 data points which is generated from Eq. (20).

$$f(x) = 3 + \frac{7(\sin x)(\ln x)^x}{x^{\ln x}} + \varepsilon \ , \tag{20}$$

Where ε is uniform random number that is generated in internal $\varepsilon \in [-0.5, 0.5]$. Three control points $\lambda_i \in [1,3,5]$ with uniform control point internals $\Delta\lambda_i = 2.0$ are considered. Fig. 3 shows the comparison of the presented model with the classic quadratic B-spline method, as explained previously, for the approximation of the above data.

The model has the preference of smoothness for approximation of data and closeness to the real values.

Problem 2.

For this problem about 53 data pairs are chosen from the following equation,

$$f(x) = 16\, Sin\ (\frac{\pi}{2.3}\sqrt{1.33\,x}) + 9\sqrt{2}\, e^{-1.37\,(x-4.201)^2} + \varepsilon \tag{21}$$

Where $x \in [0,5]$ and $\varepsilon \in [-2,2]$ is uniform random number. The control points are $\lambda_i \in \{0,1,2,3,4,5\}$ with uniform increment of $\Delta\lambda_i = 1.0$. The presented model is applied for this problem. The comparison is brought in Fig. 4. Also the quadratic B-spline curves are in the figure. As it can be seen from the figure, the presented curve is smooth and closer to the real solution $f(x)$. If two more control points $(\lambda_{-1}, \lambda_6)$ are added for reduction the effect of boundary conditions at the first and last control points the accuracy of model curve will be improved as Fig. 5.

Problem 3.

The same model formulation was applied to the problem where its function is as follows:

$$f(x) = 1 - e^{-\frac{x^2}{15}} + 7\,e^{-\frac{(x-4)^2}{3}} + \varepsilon \ . \tag{22}$$

Where $\varepsilon \in [-1,1]$ and $x \in [0,10]$. About 101 data pairs are chosen with uniform increment $\Delta x_i = 0.1$. The control points λ_i are uniformly distributed on the domain x, $\lambda_i \in \{0,2,4,6,8,10\}$ and $\Delta\lambda_i = 2.0$. The results are on Fig. (6). As it can be seen the model presents a perfect approximation curve with respect to the data generated from the function $f(x)$ and the general B-spline approximation method doesn't present a smooth curve. The SSE (sum of square error) for the model curve with respect to function $f(x)$ is SSE=0.532.

Problem 4.

This problem with respect to the variation and gradient has a complex curve. The 70 data points are generated from the following function,

$$f(x) = (-0.5\,e^{\frac{3x}{4}} + 5.4364\,e^{\frac{x}{2}})\, Sin\, x^{\frac{1}{3}} + \varepsilon \ . \tag{23}$$

Where $x \in [0,9.6]$ and $\varepsilon \in [-1,1]$. Δx_i's are not uniform but have an internal of $\Delta x_i \in [0.1-0.2]$. There are 10 control points $\lambda_0, \lambda_1, \ldots, \lambda_{10}$ that are not uniformly distributed on domain x including to boundaries control points $\lambda_i \in [-1.6, 0, 3, 4, 5, 6.4, 7.7, 8.8, 9.6, 10.4]$. The data, model curve and function $f(x)$ are shown in Fig. 7 for comparison. The results show a suitable compatibility and a reasonable relationship between the model curve and the generating function $f(x)$ for the governing set of data.

Conclusion

In engineering practice the usage and application of a set of huge data is common. We somehow need to approximate them by some curves. The presented model has applicability to estimate and approximate the data pairs with highly variations and highly volumes. The

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number of control points, their positions and increments define the suitability of the curve. They should be selected in a way that model curve pursues all global variations of data. The control point increments could be uniform or non-uniform, for the latter case some transformations should be considered and included in the formulation. The results show the increasing of control points or applying two more ones for the effects of domain boundaries cause an improvement to the approximation model curve.

Fig. 1. B-spline function.

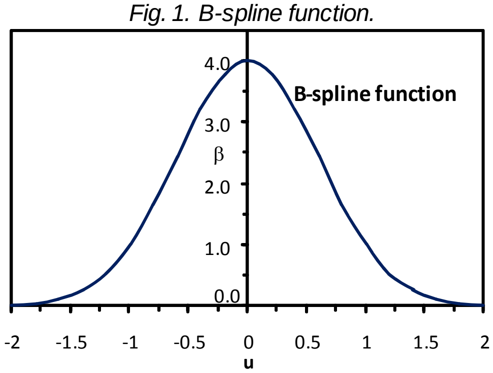


Fig. (2), heptadiagonal matrix of the linear system.

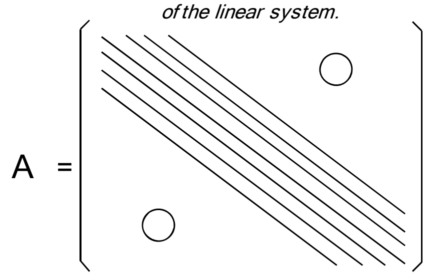


Fig. 3. The approximation of dat with the model curve.

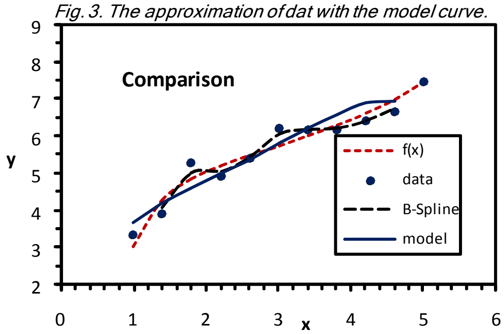


Fig. 4. The comparison between the model and real curves.

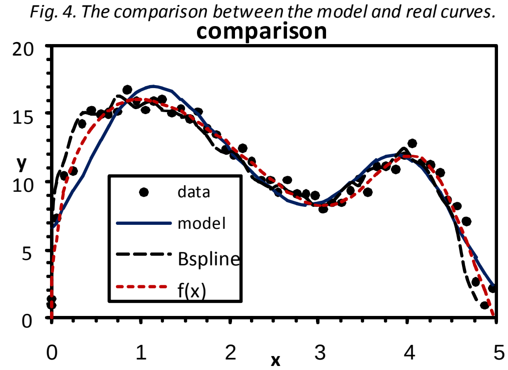


Fig. 5. The model curve with the effect of boundary conditions.

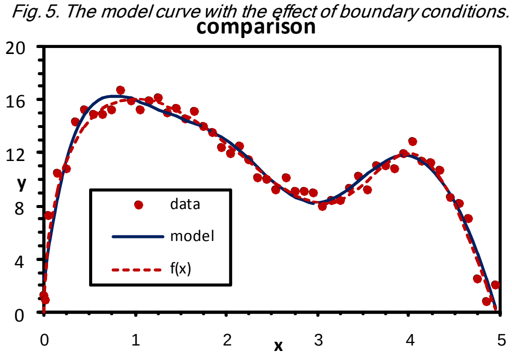


Fig. 6. The comparison between the model and real curves.

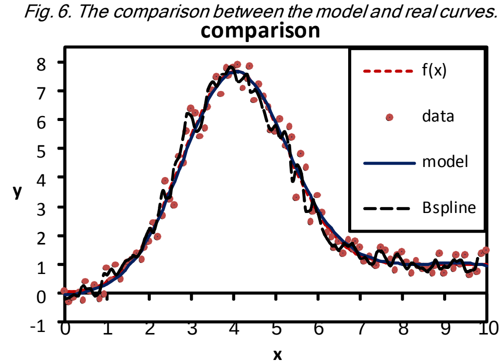
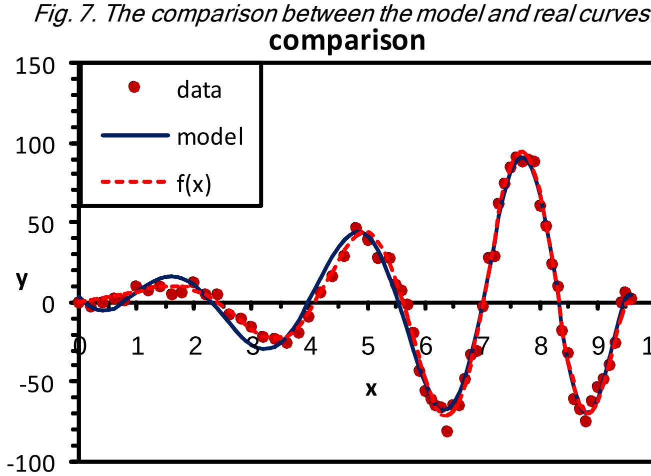


Fig. 7. The comparison between the model and real curves.



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