Indian Journal of Science and Technology Vol. 4 No. 5 (May 2011) ISSN: 0974- 6846

A generalization of Wolstenholme's Theorem

Yali Su[1] Jiming Yang[2] and Shuying Li[3]

[1]School of Information Technology and Engineering,Yuxi Normal University, Yuxi,Yunnan ,China- 653100.
[2]School of Science, Yuxi Normal University, Yuxi,Yunnan ,China- 653100.
[3]College of Resource and Environment, Yuxi Normal University, Yuxi,Yunnan ,China- 653100.

yxshuyali@yahoo.com.cn

Abstract

A generalization of Wolstenholme's Theorem is given.

Keywords: Wolstenholme's Theorem; prime number; residue

Introduction

In number theory, there is a well-known theorem about prime numbers, it is Wolstenholme Theorem. Let prime number $p>3$, Let $\frac{1}{s}$ satisfy:

$$ss^*\equiv 1\left(\bmod p^2\right)$$

s^* is an integer,then

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-1}\equiv 0\left(\bmod p^2\right). \qquad (1)$$

This theorem lead to a number of Chinese and foreign scholars to study it with a strong interest. It has been generalized to many forms.

Zhang shusheng(1989) pointed out generalization of Wolstenholme's Theorem of the following Proposition 1.

Proposition 1 Let prime number $p>3$, Let $\frac{1}{s}$ satisfy:

$$ss^*\equiv 1\left(\bmod p^2\right)$$

s^* is an integer,then

$$\sum_{k=1}^{p-1}\frac{1}{tp+k}\equiv 0\left(\bmod p^2\right) \qquad (2)$$

In the expressions, t is any integer.

Chen Keyin(2005) pointed out the following Proposition 2, which is equivalent with the theorem of Wolstenholme.

Proposition 2 Let prime number $p>3$, then

$$\sum_{k=1}^{p-1}k^{p(p-1)-1}\equiv 0\left(\bmod p^2\right). \qquad (3)$$

At the same time, Chen Keyin also made the promotion of Proposition 2 and obtain the following Proposition 3.

Proposition 3 Let prime number $p>3$, n is an odd number greater than 1, $p-1$ is not divisible by $n-1$,t is any integer,then

$$\sum_{k=1}^{p-1}\left(tp+k\right)^n\equiv 0\left(\bmod p^2\right). \qquad (4)$$

Zeng denggao(2004) pointed out generalization of Wolstenholme's Theorem, it is the following Proposition 4.

Proposition 4 p is a prime number

$$p>3,\nu,t_0,t_k\in Z\left(1\le k\le p-1\right)$$

$$0\le 2\nu<p-3,\left(t_0,p\right)=1$$

$$t_1+t_{p-1}=t_2+t_{p-2}=\cdots=t_{\frac{p-1}{2}}+t_{\frac{p-1}{2}}.$$

s^* is an integer and Let s^* satisfy:

$$ss^*\equiv 1\left(\bmod p^2\right)\text{,then}$$

$$\sum_{k=1}^{p-1}\overline{\left(t_kp+kt_0\right)^{2\nu+1}}\equiv 0\left(\bmod p^2\right). \qquad (5)$$

Hong Shaofang(2007) mentioned generalization of Wolstenholme's Theorem, it is the following Proposition 5.

Proposition 5 Let m and n is any integer , $m\ge 0, n\ge 1$, $\langle n\rangle=\{1,\cdots,n\}$. $p_1,\cdots,p_n$ are different prime numbers and all greater than 3,then

$$\sum_{\substack{j=1\\ \forall i\in\langle n\rangle,(j,p_i)=1}}^{p_1\cdots p_n}\frac{1}{mp_1\cdots p_n+j}. \qquad (6)$$

In the expressions, molecules of fraction can be divisible by $p_1^{\ 2}\cdots p_n^{\ 2}$.

New generalization of Wolstenholme's Theorem

This paper describes another generalized form of Wolstenholme's Theorem, It is the following theorem.

Theorem Let prime number $p>3$, t is any integer, r is a non-negative integer , $p^r \mid 2t+1$.Let $\frac{1}{s}$ satisfy:

$$ss^*\equiv 1\left(\bmod p^{r+2}\right)$$

s^* is an integer,then

$$\sum_{k=1}^{p-1}\frac{1}{tp+k}\equiv 0\left(\bmod p^{r+2}\right). \qquad (7)$$

when $t=0, r=0$, it is Wolstenholme theorem.

Proof

Lemma Let prime number $p>3$, then

$$\sum_{i=1}^{\frac{p-1}{2}}\prod_{\substack{j=1\\ j\neq i\\ j\neq p-i}}^{p-1} j\equiv 0\left(\bmod p\right). \qquad (8)$$

Proof (1) multiplied by $(p-1)!$ on both sides, can be get:

$$\sum_{i=1}^{p-1}\prod_{\substack{j=1\\ j\neq i}}^{p-1} j\equiv 0\left(\bmod p^2\right). \qquad (9)$$

But

$$\sum_{i=1}^{p-1}\prod_{\substack{j=1\\ j\neq i}}^{p-1} j=\sum_{i=1}^{\frac{p-1}{2}}\left[\prod_{\substack{j=1\\ j\neq i}}^{p-1} j+\prod_{\substack{j=1\\ j\neq p-i}}^{p-1} j\right]$$

$$=\sum_{i=1}^{\frac{p-1}{2}}\left[(p-i)\prod_{\substack{j=1\\ j\neq i\\ j\neq p-i}}^{p-1} j+i\prod_{\substack{j=1\\ j\neq i\\ j\neq p-i}}^{p-1} j\right]=p\sum_{i=1}^{\frac{p-1}{2}}\prod_{\substack{j=1\\ j\neq i\\ j\neq p-i}}^{p-1} j. \qquad (10)$$

By (9), (10), we can get (8) holds.

Easy to obtain (11) from the proof of the theorem.

$$\sum_{i=1}^{p-1}\prod_{\substack{j=1\\ j\neq i}}^{p-1}(tp+j)=\sum_{i=1}^{\frac{p-1}{2}}\left[\prod_{\substack{j=1\\ j\neq i}}^{p-1}(tp+j)+\prod_{\substack{j=1\\ j\neq p-i}}^{p-1}(tp+j)\right]$$

$$=\sum_{i=1}^{\frac{p-1}{2}}\left[(tp+p-i)\prod_{\substack{j=1\\ j\neq i\\ j\neq p-i}}^{p-1}(tp+j)+(tp+i)\prod_{\substack{j=1\\ j\neq i\\ j\neq p-i}}^{p-1}(tp+j)\right] \qquad (11)$$

$$=p\left(2t+1\right)\sum_{i=1}^{\frac{p-1}{2}}\prod_{\substack{j=1\\ j\neq i\\ j\neq p-i}}^{p-1}(tp+j).$$

Obtained (12) by the Lemma.

$$\sum_{i=1}^{\frac{p-1}{2}}\prod_{\substack{j=1\\ j\neq i\\ j\neq p-i}}^{p-1}(tp+j)$$

$$\equiv\sum_{i=1}^{\frac{p-1}{2}}\prod_{\substack{j=1\\ j\neq i\\ j\neq p-i}}^{p-1} j\equiv 0\left(\bmod p\right). \qquad (12)$$

From the known conditions $p^r \mid 2t+1$,(11)and (12), we can get:

$$\sum_{i=1}^{p-1}\prod_{\substack{j=1\\ j\neq i}}^{p-1}(tp+j)\equiv 0\left(\bmod p^{r+2}\right). \qquad (13)$$

By (13), can be get:

$$\left(\prod_{j=1}^{p-1}(tp+j)\right)\sum_{i=1}^{p-1}(tp+i)^*=\sum_{i=1}^{p-1}(tp+i)^*\prod_{j=1}^{p-1}(tp+j)$$

$$\equiv\sum_{i=1}^{p-1}\prod_{\substack{j=1\\ j\neq i}}^{p-1}(tp+j)\equiv 0\left(\bmod p^{r+2}\right). \qquad (14)$$

But $\left(\prod_{j=1}^{p-1}(tp+j),\ p^{r+2}\right)=1$, by (14), we can get:

$$\sum_{i=1}^{p-1}(tp+i)^*\equiv 0\left(\bmod p^{r+2}\right). \qquad (15)$$

By (15), we can get (7) holds.

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