

## Felicitious labeling in Extended Duplicate Graph of Twig $T_m$

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### Abstract

In this paper, we show that the class of Extended Duplicate Graph of a Twig is Felicitious.

**Keywords:** Graph labeling, Graceful labeling, Felicitious labeling, Duplicate Graphs, Twigs.

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### Introduction

The concept of graph labeling was introduced by Rosa (1967). A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs serve as useful models for a broad range of applications such as circuit design, radar, astronomy, coding theory, communication network addressing and data base management (Gallian, 2010). Hence, in the intervening years various labeling of graphs such as graceful labeling, harmonious labeling, magic labeling, antimagic labeling, bimagic labeling, prime labeling, cordial labeling, mean labeling, arithmetic labeling etc., have been studied in over 1100 papers (Gallian, 2010).

The study of graceful graphs and graceful labeling methods was introduced by Rosa (1967). Rosa defined a  $\beta$  - valuation of a graph  $G$  with  $e$  edges as an injection from the vertices of  $G$  to the set  $\{0,1,2,\dots,e\}$  such that when each edge  $xy$  is assigned the label  $|f(x) - f(y)|$  the resulting edge labels are distinct.  $\beta$  - valuations are functions that produce graceful labelings.

However, the term graceful labeling was not used until Golomb (1972) studied such labelings several years later.

An injective function from the vertices of a graph  $G$  with  $q$  edges to the set  $\{0, 1, \dots, q\}$  is called felicitious if the edge labels induced by  $\{f(x) + f(y)\} \pmod{q}$  for each edge  $xy$  are distinct. This definition first appeared in a paper by Lee *et al.* (1991) and is attributed to Balakrishnan and Kumar (1994) proved the conjecture of Lee *et al.* (1991) that every graph is a subgraph of a felicitious graph by showing the stronger result that every graph is a subgraph of a sequential graph. Among the graphs known to be felicitious are:  $C_n$  except when  $n = 2 \pmod{4}$ ;  $K_{m,n}$  when  $m, n > 1$ ; Lee *et al.* (1991) conjectured that the  $n$ -cube is felicitious. Balakrishnan *et al.* (1996) obtained numerous results on felicitious labelings. Yegnanarayanan (1999) conjectures that the graphs obtained from an even cycle by attaching  $n$  new vertices to each vertex of the cycle is felicitious. In this paper, we prove that the Extended Duplicate Graph of Twigs are Felicitious.

### Preliminaries

In this section we give the basic notions relevant to this paper. Let  $G = G(V, E)$  be a finite, simple and undirected graph with  $p$  vertices and  $q$  edges. By a labeling we mean a one-to-one mapping that carries a set of graph elements onto a set of numbers, called labels. In this paper we deal labeling with domain as the set of all vertices.

**Definition (Graceful) :** A function  $f$  is said to be Graceful of a graph  $G$  with ' $q$ ' edges if  $f$  is 1-1 from  $V \rightarrow \{0,1,2,3,\dots,q\}$  such that for each edge  $uv$  is assigned the label  $|f(u) - f(v)|$ , the resulting edge labels are distinct numbers  $\{1,2,3,\dots,q\}$ .

**Definition (Felicitious) :** An injective function  $f : V \rightarrow \{0,1,2,\dots,q\}$  where  $|E| = q$ , is called felicitious, if the edge labels induced by  $\{f(x) + f(y)\} \pmod{q}$  for each edge  $xy$  are distinct.

**Definition (Twig) :** A graph  $G(V, E)$  obtained from a path by attached exactly two pendent edges to each internal vertices of the path is called a Twig graph. Generally, a Twig  $T_m$  with  $m$  internal vertices, has  $3m+1$  edges and  $3m+2$  vertices. We classify the Twigs into two categories such as  $T_{2n}$  and  $T_{2n-1}$ ,  $n \in \mathbb{N}$ .

**Definition** Let  $G(V, E)$  be a simple graph. A duplicate graph of  $G$  is  $DG = (V_1, E_1)$  where the vertex set  $V_1 = V \cup V'$  and  $V \cap V' = \phi$  and  $f : V \rightarrow V'$  is bijective, (for  $v \in V$ , we write  $f(v) = v'$  for convenience) and the edge set  $E_1$  of  $DG$  is defined as follows: The edge  $ab$  is in  $E_1$  if and only if both  $ab'$  and  $a'b$  are edges in  $E$ . Clearly the duplicate graph of the twig graph is disconnected. In order to view as a connected graph we give the following definition from Selvam *et al.* (2010, 2011).

**Definition EDG ( $T_m$ ):** Let  $G(V, E)$  be a Twig graph  $T_m$  and let  $DG = (V_1, E_1)$  where the vertex set  $V_1 = V \cup V'$  and  $V \cap V' = \phi$  be a duplicate graph of a Twig  $T_m$ . Add an edge between any one vertex from  $V$  to any other vertex in  $V'$ , except the terminal vertices of  $V$  and  $V'$ . For convenience, let us take  $v_2 \in V$  and  $v'_2 \in V'$  and thus the edge  $(v_2, v'_2)$  is formed. We call this new graph as the

Extended Duplicate Graph of the Twig  $T_m$  and it is denoted as  $EDG(T_m)$ . Clearly this  $EDG(T_m)$  has  $(6m+4)$  vertices and  $(6m+3)$  edges.

### Results

In this section, we present an algorithm and prove the existence of felicitous labeling for  $EDG(T_m)$ .

### Algorithm:

**Input:**  $EDG(T_m)$  with  $6m+4$  vertices and  $6m+3$  edges.

**Step 1 :** Denote the  $6m+4$  vertices as

$$V = \{v_1, v_2, \dots, v_{3m+1}, v_{3m+2}, v'_1, v'_2, \dots, v'_{3m+1}, v'_{3m+2}\}$$

**Step 2 :** If 'm' is even, then  $m = 2n$ ;  $n \in \mathbb{N}$

If  $n \in \mathbb{N}$ , then the  $EDG(T_m)$  is of the form  $EDG(T_{2n})$ ;  
 $n \in \mathbb{N}$

Define  $f: V \rightarrow \{0, 1, 2, 3, \dots, 6m+3\}$  such that, for

$$0 \leq i \leq \frac{m-2}{2}$$

$$(i). f(v_{6i+3}) = 3(m-i), f(v_{6i+4}) = 3(m-i) - 1, \\ f(v_{6i+5}) = 3(m-i) - 2$$

$$(ii). f(v_{6i+6}) = 3i + 1, f(v_{6i+7}) = 3i + 2, f(v_{6i+8}) = 3i + 3$$

$$(iii). f(v'_{6i+3}) = 3(m+i) + 3, f(v'_{6i+4}) = 3(m+i) + 4, f(v'_{6i+5}) = 3(m+i) + 5$$

$$(iv). f(v'_{6i+6}) = 3(2m-i) + 2, f(v'_{6i+7}) = 3(2m-i) + 1, f(v'_{6i+8}) = 3(2m-i)$$

and also

$$f(v_1) = 3m + 1, f(v_2) = 0, f(v'_1) = 3m + 2, f(v'_2) = 6m + 3$$

### Step 3:

If  $m$  is odd, then  $m = 2n-1$ ;  $n \in \mathbb{N}$

The  $EDG(T_m)$  is of the form  $EDG(T_{2n-1})$ ;  $n \in \mathbb{N}$

Define  $f: V \rightarrow \{0, 1, 2, 3, \dots, 6m+3\}$  such that  $f(v_i)$  is same

as in (i) to (iv) of **Step 2** with the limits  $0 \leq i \leq \frac{m-1}{2}$ ,

$$0 \leq i \leq \frac{m-3}{2}, \quad 0 \leq i \leq \frac{m-1}{2} \quad \text{and} \quad 0 \leq i \leq \frac{m-3}{2}$$

respectively.

### Step 4:

Define  $f^*: E \rightarrow \mathbb{N}$  such that

$$f^*(v_i, v_j) = \{f(v_i) + f(v_j)\} \pmod{q}$$

**Output :** Labeled  $EDG(T_m)$

### Theorem:

The Extended Duplicate Graph of Twig  $T_m$ ,  $m \geq 1$  is felicitous.

### Proof:

Let  $EDG(T_m)$  be a Extended duplicate graph of the Twig  $T_m$ . Clearly

$EDG(T_m)$  has  $6m+4$  vertices and  $6m+3$  edges.

Denote the set of vertices as

$$V = \{v_1, v_2, \dots, v_{3m+1}, v_{3m+2}, v'_1, v'_2, \dots, v'_{3m+1}, v'_{3m+2}\}$$

### Case (1)

Let  $T_m$  be a Twig where  $m = 2n$ ,  $n \in \mathbb{N}$ .

Consider the Twig of the Type  $T_{2n}$ ,  $n \in \mathbb{N}$ .

To label the vertices, define a map  $f: V \rightarrow \{0, 1, 2, 3, \dots, 6m+3\}$  as given in **Step 2** of the above algorithm.

The vertices  $v_1$  and  $v_2$  are labeled as  $3m+1$  and  $0$ ;

the vertices  $(v_3, v_4, v_5)$ ,  $(v_9, v_{10}, v_{11})$  .....

$(v_{3m-3}, v_{3m-2}, v_{3m-1})$ ,  $(v_{3m+2}, v_{3m+1}, v_{3m})$ ,

$(v_{3m-4}, v_{3m-5}, v_{3m-6})$ .....  $(v_8, v_7, v_6)$  receive consecutive

numbers such as  $3m, 3m-1, 3m-2, \dots, 3, 2, 1$  as labels; the

vertices  $v'_1$  and  $v'_2$  are labeled as  $3m+2$  and  $6m+3$ ; the

vertices  $(v'_3, v'_4, v'_5)$ ,  $(v'_9, v'_{10}, v'_{11})$  .....

$(v'_{3m-3}, v'_{3m-2}, v'_{3m-1})$ ,  $(v'_{3m+2}, v'_{3m+1}, v'_{3m})$ ,

$(v'_{3m-4}, v'_{3m-5}, v'_{3m-6})$ .....  $(v'_8, v'_7, v'_6)$  receive consecutive

numbers  $3m+3, 3m+4, 3m+5, \dots, 6m, 6m+1, 6m+2$  as

labels.

Thus all the  $6m+4$  vertices are labeled.

From the definition of  $EDG(T_m)$ , the  $6m+3$  edges of

$EDG(T_m)$  are of the form  $(v_2, v'_2)$ ,  $(v_1, v'_2)$ ,  $(v_2, v'_1)$ ,

$(v_{2+3i}, v'_{3i+3+j})$ ,  $(v'_{2+3i}, v_{3i+3+j})$  for

$$0 \leq j \leq 2, 0 \leq i \leq m-1.$$

The induced function  $f^*: E \rightarrow \mathbb{N}$  is defined such that

$$f^*(v_i, v_j) = \{f(v_i) + f(v_j)\} \pmod{q}$$

Thus the edges  $(v'_1, v_2)$ ,  $(v_2, v'_3)$ ,  $(v_2, v'_4)$ ,  $(v_2, v'_5)$ ,

$(v'_5, v_6)$ ,  $(v'_5, v_7)$ ,  $(v'_5, v_8)$  .....  $(v'_{3m-1}, v_{3m})$ ,

$(v'_{3m-1}, v_{3m+1})$ ,  $(v'_{3m-1}, v_{3m+2})$  receive consecutive

integers such as  $3m+2, 3m+3, 3m+4, 3m+5, \dots, 6m, 6m+1,$

$6m+2$  as labels; the edge  $(v_2, v'_2)$  receive the number  $0$

as label; the edges  $(v'_{3m+2}, v_{3m-1})$ ,  $(v'_{3m+1}, v_{3m-1})$ ,

$(v'_{3m}, v_{3m-1})$ ,

Fig. 1. Felicitous labeling EDG  $T_6$

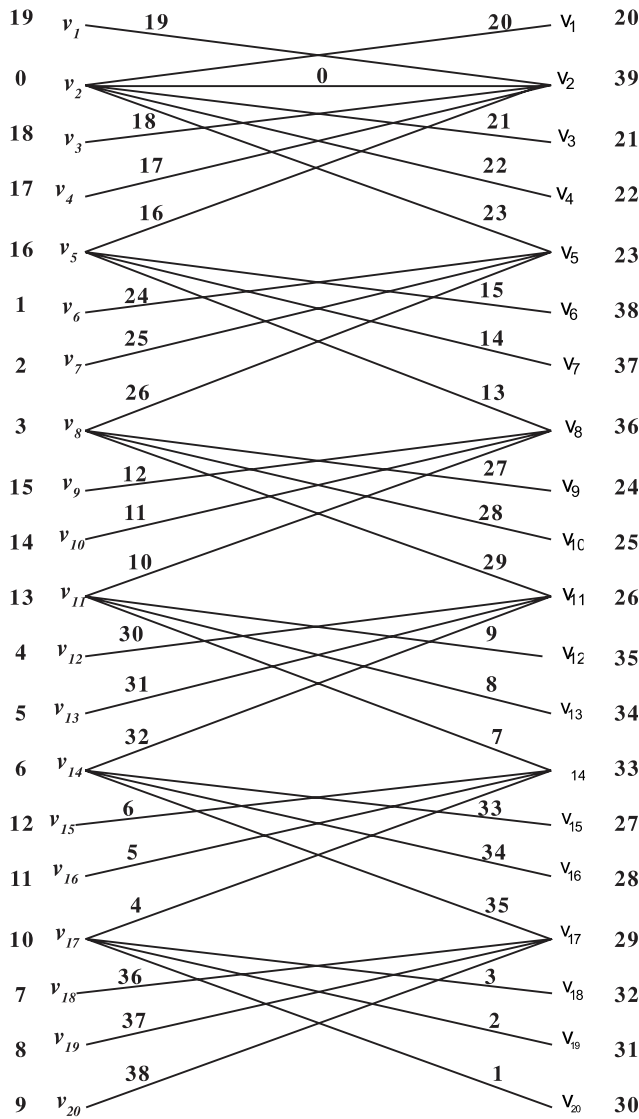
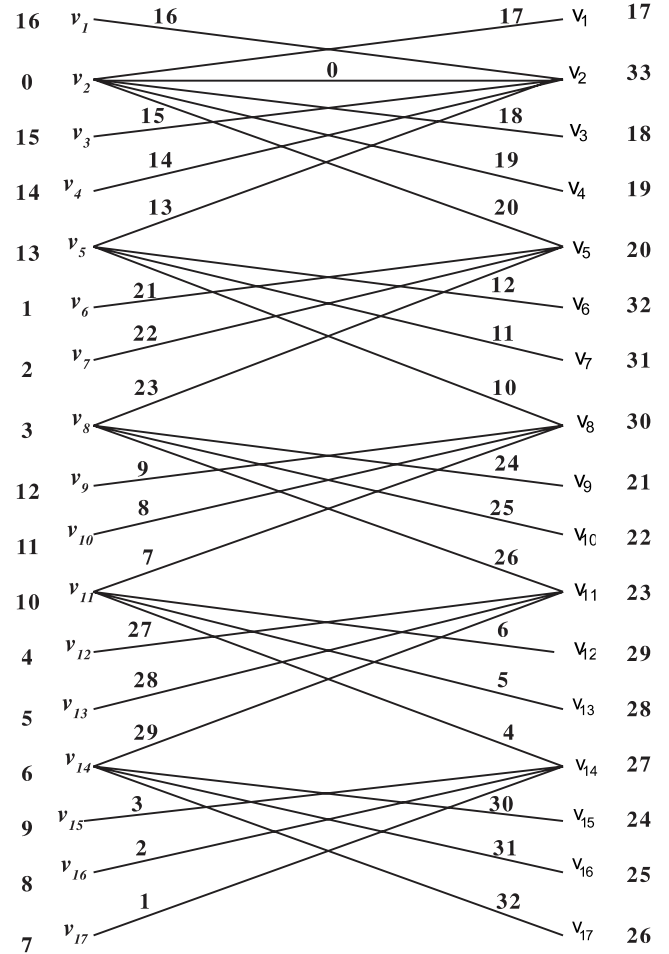


Fig. 2. Felicitous labeling EDG  $T_5$



$(v_{3m-1}, v'_{3m-4}), (v_{3m-2}, v'_{3m-4}), (v_{3m-3}, v'_{3m-4}) \dots (v_5, v'_2), (v_4, v'_2), (v_3, v'_2), (v_2, v'_1)$  receive consecutive integers such as 1,2,3,..... $3m-2, 3m-1, 3m, 3m+1$  as labels. Thus all the edges are labeled as  $\{0,1,2,3,4, \dots, 6m+2\}$  and all are distinct. (The whole idea of the proof is illustrated in Fig(2)).

Hence  $EDG(T_m)$  where  $m=2n, n \in \mathbb{N}$  is felicitous.

**Case 2:**

Let  $T_m$  be a Twig where  $m=2n-1, n \in \mathbb{N}$ .

Consider the Twig of the Type  $T_{2n-1}, n \in \mathbb{N}$ .

To label the vertices, define a map  $f:V \rightarrow \{0,1,2,3, \dots, 6m+3\}$  as given in **Step 3** of the above algorithm.

The vertices  $v_1$  and  $v_2$  are labeled as  $3m+1$  and  $0$ ; the vertices  $(v_3, v_4, v_5), (v_9, v_{10}, v_{11}) \dots \dots \dots (v_{3m}, v_{3m+1}, v_{3m+2}), (v_{3m-1}, v_{3m-2}, v_{3m-3}) \dots \dots \dots (v_8, v_7, v_6)$  receive consecutive integers such as  $3m, 3m-1, 3m-2, \dots, 3, 2, 1$  as labels; the vertices  $v'_1$  and  $v'_2$  are labeled as  $3m+2$  and  $6m+3$ ; the vertices  $(v'_3, v'_4, v'_5), (v'_9, v'_{10}, v'_{11}) \dots \dots \dots (v'_{3m}, v'_{3m+1}, v'_{3m+2}),$

$(v'_{3m-1}, v'_{3m-2}, v'_{3m-3}), \dots \dots \dots (v'_8, v'_7, v'_6)$  receive consecutive integers such as  $3m+3, 3m+4, 3m+5, \dots, 6m, 6m+1, 6m+2$  as labels.

Thus all the  $6m+4$  vertices are labeled.

From the definition of  $EDG(T_m)$ , the  $6m+3$  edges of  $EDG(T_m)$  are of the form  $(v_2, v_2')$ ,  $(v_1, v_2')$ ,  $(v_2, v_1')$ ,  $(v_{2+3i}, v_{3i+3+j}')$ ,  $(v_{2+3i}', v_{3i+3+j})$  for  $0 \leq j \leq 2$ ,  $0 \leq i \leq m-1$ .

The induced function  $f^* : E \rightarrow N$  is defined such that

$$f^*(v_i, v_j) = \{f(v_i) + f(v_j)\} \pmod{q}$$

Thus the edges  $(v_1', v_2)$ ,  $(v_2, v_3')$ ,  $(v_2, v_4')$ ,  $(v_2, v_5')$ ,  $(v_5, v_6')$ ,  $(v_5, v_7')$ ,  $(v_5, v_8')$  .....  $(v_{3m-1}', v_{3m})$ ,  $(v_{3m-1}', v_{3m+1})$ ,  $(v_{3m-1}', v_{3m+2})$  receive consecutive integers such as  $3m+2, 3m+3, 3m+4, 3m+5, \dots, 6m, 6m+1, 6m+2$  as labels; the edge  $(v_2, v_2')$  receive the number 0 as label; the edges  $(v_{3m+2}', v_{3m-1})$ ,  $(v_{3m+1}', v_{3m-1})$ ,  $(v_{3m}', v_{3m-1})$ ,  $(v_{3m-1}', v_{3m-4})$ ,  $(v_{3m-2}', v_{3m-4})$ ,  $(v_{3m-3}', v_{3m-4})$  .....  $(v_5, v_2')$ ,  $(v_4, v_2')$ ,  $(v_3, v_2')$ ,  $(v_2, v_1')$  receive consecutive integers such as  $1, 2, 3, \dots, 3m-2, 3m-1, 3m, 3m+1$  as labels.

Thus all the edges are labeled as  $\{0, 1, 2, 3, 4, \dots, 6m+2\}$  and all are distinct. (The above concept is illustrated in Fig(1)).

Hence  $EDG(T_m)$  where  $m=2n$ ,  $n \in \mathbb{N}$  is felicitious.

Hence the proof of the theorem is completed.

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