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An investigation of nonlinear KdV type equations using HPM and VIM

H. Goodarzian[1], M. Ghobadi[2], M.A. Farahabadi[3], H. Mohammadnezhad[4] and S.S. Hejazi[1]

[1] Young Researchers Club, *Islamic Azad University- Jouybar Branch, Jouybar, Iran*
[2]*Sama Technical and Vocational Training College, Islamic Azad University, Sari Branch, Sari, Iran*
[3]*Islamic Azad University- Qaemshahr Branch, Qaemshahr, Iran*
[4]*Islamic Azad University- Sari branch, Sari, Iran*
[1]Young Researchers Club, *Islamic Azad University, Jouybar Branch, Jouybar, Iran*
hamed_goodarzian@yahoo.com

Abstract

The KdV equation appears in fluid mechanics. This equation incorporates both convection and diffusion in fluid dynamics, and is used to describe the structure of shock waves. The objective of this paper is to present a comparative study of He's Homotopy perturbation method (HPM) and variational iteration method (VIM) for the semi analytical solution of three different Kortweg-de Vries (KdV) type equations called KdV, K(2,2,) and modified KdV (Burgers) equations. The study has been highlighted the efficiency and capability of aforementioned methods in solving these nonlinear problems which has been arisen from a number of important physical phenomenon.

Keywords: Variational Iteration Method (VIM); Homotopy Perturbation Method (HPM); KdV Equation; Modified KdV Equation.

Introduction

It was 1895 that Korteweg and Vries derived KdV equation to model Russell's phenomenon of solitons (Korteweg & Vries, 1895) like shallow water waves with small but finite amplitudes. Solitons are localized waves that propagate without change of its shape and velocity properties and stable against mutual collision (Khattak & ul-Islam S, 2008). It has also been used to describe a number of important physical phenomena such as magneto hydrodynamics waves in warm plasma, acoustic waves in an inharmonic crystal and ion-acoustic waves (Ozis & Ozer, 2006)

Consider three models of KdV equation called KdV, K (2, 2) and modified KdV (Korteweg & Vries, 1895) as given respectively by:

$$u_t - 3(u^2)_x + u_{xxx} = 0, \qquad (1.1)$$

$$u_t + (u^2)_x + (u^2)_{xxx} = 0, \qquad (1.2)$$

and

$$u_t + \frac{1}{2}(u^2)_x - u_{xx} = 0. \qquad (1.3)$$

Eq. (1.1) is the pioneering equation that gives rise to solitary wave solutions. Solitons, which are waves with infinite support, are generated as a result of the balance between the nonlinear convection $(u^n)_x$ and the linear dispersion u_{xxx} in the above equations. Solitons are localized waves that propagate without change of their shape and velocity properties and stable against mutual collisions (Abdou & Soliman, 2005).

The equation of *K*(*n,n*) (Khattak & ul-Islam S, 2008):

$$u_t + (u^n)_x + (u^n)_{xxx} = 0, \qquad (1.4)$$

is the pioneering equation for compactons. In solitary wave's theory, compactons are defined as solitons with finite wave lengths or solitons free of exponential tails. Compactons are generated as a result of the delicate interaction between nonlinear convection $(u^n)_x$ with the genuine nonlinear dispersion $(u^n)_{xxx}$ in (1.4).

Finally, the modified KdV equation appears in fluid mechanics. This equation incorporates both convection and diffusion in fluid dynamics, and is used to describe the structure of shock waves (Aboulvafa *et al*., 2006). Hence, KdV type equations have significant roles in engineering and physics. Besides, the analytical solutions of these governing equations may guide authors to know the described process deeply and sometimes leads them to know some facts that are not simply understood through common observations, but it is quite difficult to obtain the analytical solution of these problems as these are functioning highly nonlinear.

Many researchers studied the similar kinds of problems in other applications (Tolou *et al*., 2008; Ganji, 2006; Zahedi & Okazi, 2010; He, 2006a, 2006b, 2005) and many powerful methods have been proposed to seek the exact solutions of nonlinear differential equations; for instance, Darboux transformation (Wadati *et al*., 1975), the inverse scattering method (Gardner *et al*., 1967), the tanh method (Soliman, 2006), the sine-cosine method (Yan & Zhang, 2000), the homogeneous balance method (Yan & Zhang, 2001), and the Riccati expansion method with constant coefficients (Yan, 2001).

In this paper, He's variational iteration method (VIM) and homotopy perturbation method (HPM) and Liao's homotopy analysis method (HAM) are used to conduct an analytic study on the KdV, the K(2,2) and the modified KdV equations in order to show all the methods above, are capable in solving a large number of linear or nonlinear differential equations, also all the aforementioned methods give rapidly convergent successive approximations of the exact solution if such solution exists, otherwise approximations can be used for numerical purposes.

He's Variational Iteration method

Fundamental

To illustrate the basic concepts of variational iteration method, we consider the following deferential equation (He, 2006b):

$$Lu + Nu = g(x) \quad (2.1)$$

where L is a linear operator, N a nonlinear operator, and $\mathrm{g(x)}$ a heterogeneous term. According to VIM, we can construct a correction functional as follows (He , 2006b):

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{L\, u_n(\tau) + N\, \tilde{u}_n(\tau) - g(\tau)\}d\tau \quad (2.2)$$

where λ is a general Lagrangian multiplier (He , 2006a), which can be identified optimally via the variational theory, the subscript n indicates the n^{th} order approximation, $\tilde{u}_n$ which is considered as a restricted variation, i.e. $\delta\, \tilde{u}_n = 0$.

The Application

Example: Considering the KdV equation as:

$$u_t - 3(u)^2{}_x + u_{xxx} = 0, \quad -\infty < x < +\infty, \quad t > 0 \quad (2.3)$$

with the following initial condition:

$$u(x,0) = 6x \quad (2.4)$$

To solve Eqs. (2.3) and (2.4) using VIM, we have the correction functional as:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\{u_t - u_{xxt} + uu_x\}d\tau, \quad (2.5)$$

where uu_x indicates the restricted variations; i.e. $\delta(uu_x) = 0$. Making the above correction functional stationary, we obtain the following stationary conditions:

$$1 + \lambda\big|_{\tau=1} = 0 \quad (2.6.a)$$

$$\lambda' = 0 \quad (2.6.b)$$

The Lagrangian multiplier can therefore be identified as:

$$\lambda = \text{-}1 \quad (2.7)$$

substituting Eq. (2.7) into the correction functional equation system (2.5) results in the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{u_\tau - 3(u)^2{}_x + u_{xxx}\}\, d\tau \quad (2.8)$$

Each result obtained from Eq. (2.8) is $u(x,t)$ with its own error relative to the exact solution, but higher number iterations leads us to obtain results closer to the exact solution. Using the iteration formula (2.8) and the initial condition as u_0 , five iterations were made as follows ;

The first iteration results in:

$$u_1(x,t) = 6x(1+36t). \quad (2.9.a)$$

The second iteration results in:

$$u_2(x,t) = 6x(1+36t+1296t^2+15552t^3). \quad (2.9.b)$$

And finally, the fifth iteration results in:

$$u_5(x,t) = 6x(1+36\,t+1296\,t^2+15552\,t^3+1679616t^4+60466176t^5) + small\ terms. \quad (2.9.c)$$

It is obvious that $u_n(x,t)$ converges to $\frac{6x}{1-36t}$ as an exact solution for Eqs. (2.3, 2.4).

Example: We consider the K(2,2) equation

$$u_t + (u)^2{}_x + (u^2)_{xxx} = 0, \quad x \in R,\ t > 0, \quad (2.10)$$

$$u(x,0) = x. \quad (2.11)$$

To solve Eqs. (2.10-2.11) using VIM, we have the correction functional as:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\{u_\tau + (u)^2{}_x + (u^2)_{xxx}\}\, d\tau. \quad (2.12)$$

Making the above correction functional stationary, we obtain the following stationary conditions:

$$1 + \lambda\big|_{\tau=1} = 0, \quad (2.13.a)$$

$$\lambda' = 0. \quad (2.13.b)$$

The Lagrangian multiplier can therefore be identified as:

$$\lambda = \text{-}1. \quad (2.14)$$

Substituting Eq. (2.14) into the correction functional equation system (2.12) results in the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{u_\tau + (u)^2{}_x + (u^2)_{xxx}\}\, d\tau. \quad (2.15)$$

Using the iteration formula (2.15) and the initial condition as u_0 , six iterations were made as follows ;

The first iteration results in:

$$u_1(x,t) = x(1-2t), \quad (2.16.a)$$

The second iteration results in:

$$u_2(x,t) = x\left(1 - 4t + 4t^2 - \frac{8}{3}t^3\right), \quad (2.16.b)$$

And finally the sixth iteration results in:

$$u_6(x,t) = x\left(1 - 2t + 4t^2 - 8\,t^3 + 16\,t^4 - 32\,t^5 + 64\,t^6\right) + small\ terms. \quad (2.16.c)$$

Again, by trying higher iterations we can obtain the exact solution of Eqs. (2.10, 2.11) in the form of

$$u(x,t) = \frac{x}{1+2t}.$$

Example: For the third example we consider the modified $KdV(mKdV)$ equation as:

$$u_t + \frac{1}{2}\left(u^2\right)_x - u_{xx} = 0, \quad x \in R,\ t > 0, \quad (2.17)$$

$$u(x,0) = x. \quad (2.18)$$

The correction functional takes the form of:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\{u_\tau + \frac{1}{2}\left(u^2\right)_x - u_{xx}\}\, d\tau \quad (2.19)$$

where, again $\lambda = -1$. So, Eq.(2.19) changes to:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{u_\tau + \frac{1}{2}\left(u^2\right)_x - u_{xx}\} d\tau. \qquad (2.20)$$

Using the iteration formula (2.20) and the initial condition as u_0, three iterations were made and results are as follows :

$$u_1(x,t) = x(1-t), \qquad (2.21)$$

$$u_2(x,t) = x(1-t+t^2-\frac{1}{3}t^3), \qquad (2.22)$$

$$u_3(x,t) = x(1-t+t^2-t^3+\frac{2}{3}t^4), \qquad (2.23)$$

If one try the higher iterations, can have the exact solution of Eqs. (2.17, 2.18) in the form of $u(x,t) = \frac{x}{1+t}$.

He's Homotopy Perturbation Method

Fundamental

To illustrate the basic ideas of this method, we consider the following nonlinear differential Equation (He, 2006a):

$$A(u) - f(r) = 0. \quad r \in \Omega \qquad (3.1)$$

Considering the boundary conditions of:

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma \qquad (3.2)$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

The operator A can be divided into two parts of L and N, where L is the linear part, while N is a nonlinear one. Eq. (3.1) can, therefore, be rewritten as:

$$L(u) + N(u) - f(r) = 0. \qquad (3.3)$$

By the homotopy technique, we construct a homotopy as $v(r,p): \Omega \times [0,1] \rightarrow \Re$ which satisfies (He, 2006a):

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad P \in [0,1], \quad r \in \Omega \quad (3.4)$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. (3.4) which satisfies the boundary conditions. Obviously, considering Eq. (3.4) we will have:

$$H(v,0) = L(v) - l(u_0) = 0, \qquad (3.5)$$

$$H(v,1) = A(v) - f(r) = 0. \qquad (3.6)$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solution of Eq. (3.4) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots, \qquad (3.7)$$

setting $p = 1$ results in the approximate solution of Eq. (3.4):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... \qquad (3.8)$$

The combination of the perturbation method and the homotopy method is called the HPM, which lacks the limitations of the conventional perturbation methods, although this technique can have full advantages of the conventional perturbation techniques.

The series (3.8) is convergent for most cases. However, the convergence rate depends on the nonlinear operator $A(v)$. The following opinions are suggested by He (2005):

1. The second derivative of $N(v)$ with respect to v must be small because the parameter p may be relatively large, i.e. $p \to 1$.

2. The norm of $L^{-1} \partial N / \partial v$ must be smaller than one so that the series converges.

Application

Example: With the same first example as mentioned previously, the equation is as:

$$u_t - 3(u)^2{}_x + u_{xxx} = 0, \quad -\infty < x < +\infty, \quad t > 0. \qquad (3.9)$$

with the initial condition of:

$$u(x,0) = 6x. \qquad (3.10)$$

Substituting Eq. (3.9) into (3.4) and then substituting v from (3.7) and rearranging it as a power series in p, we have an equation system including $n+1$ equations to be simultaneously solved; n is the order of p in Eq. 3.7).

Assuming $n = 5$, the system is as follows:

$$\begin{cases} u_{0t} = 0, & u_0(x,0) = 6x, \\ u_{1t} - 6u_0u_{0x} + u_{0xxx} = 0, & u_1(x,0) = 0, \\ u_{2t} - 6u_1u_{0x} - 6u_0u_{1x} + u_{1xxx} = 0, & u_2(x,0) = 0, \\ u_{3t} - 6u_0u_{2x} - 6u_1u_{1x} - 6u_2u_{0x} + u_{2xxx} = 0, & u_3(x,0) = 0, \\ u_{4t} - 6u_3u_{0x} - 6u_0u_{3x} - 6u_2u_{1x} - 6u_1u_{2x} + u_{3xxx} = 0, & u_4(x,t) = 0, \\ u_{5t} - 6u_3u_{1x} - 6u_4u_{0x} - 6u_1u_{3x} - 6u_2u_{2x} - 6u_0u_{4x} + u_{4xxx} = 0 & u_5(x,t) = 0. \end{cases} .$$

One can now try to obtain a solution for equation system (3.11), in the form of:

$$u_0(x,t) = 6x, \qquad (3.12.a)$$

$$u_1(x,t) = 6x(36t), \qquad (3.12.b)$$

$$u_2(x,t) = 6x(1296\,t^2), \qquad (3.12.c)$$

$$u_3(x,t) = 6x(46656\,t^3), \qquad (3.12.d)$$

$u_4(x,t) = 6x(1679616t^4),$ (3.12.e)

$u_5(x,t) = 6x(60466176t^5).$ (3.12.f)

Having $u_i, i = 0,1,...5$, the solution $\mathrm{u}(\mathrm{x},\mathrm{t})$ is as:

$$u(x,t) = \sum_{i=0}^{5} u_i(x,t) = 6x\left[1 + 36t + (36t)^2 + (36t)^3 + (36t)^4 + (36t)^5\right]. \quad (3.13)$$

Trying higher iterations, we can obtain the exact solution of Eqs. (3.9, 3.10) in the form of $u(x,t) = \dfrac{6x}{1-36t}$.

Example: Let us consider the following equation again:

$u_t + (u)^2{}_x + (u^2)_{xxx} = 0, \quad x \in R,\ t > 0,$ (3.14)

$u(x,0) = x.$ (3.15)

Substituting Eq. (3.14) into (3.4) and then substituting v from (3.7), rearranging it as a power series in p and assuming $n = 5$, we have a system of equations including six equations to be simultaneously solved as follows:

$$\begin{cases} u_{0t} = 0, & u_0(x,0) = x \\ u_{1t} + 6u_{0x}u_{0xx} + 2u_0u_{0xxx} + 2u_0u_{0x} = 0, & u_1(x,0) = 0, \\ \begin{cases} u_{2t} + 6u_{1x}u_{0xx} + 6u_{0x}u_{1xx} + 2u_1u_{0x} \\ + 2u_0u_{1x} + 2u_1u_{0xxx} + 2u_0u_{1xxx} \end{cases} = 0, & u_2(x,0) = 0, \\ \begin{cases} u_{3t} + 6u_{2x}u_{0xx} + 6u_{0x}u_{2xx} + 6u_{1x}u_{1xx} \\ + 2u_2u_{0xxx} + 2u_0u_{2xxx} + 2u_0u_{2x} + \\ 2u_1u_{1xxx} + 2u_1u_{1x} + 2u_2u_{0x} \end{cases} = 0, & u_3(x,0) = 0, \\ \begin{cases} u_{4t} + 6u_{1x}u_{2xx} + 6u_{2x}u_{1xx} + 6u_{3x}u_{0xx} \\ + 6u_{0x}u_{3xx} + 2u_2u_{1x} + 2u_0u_{3xxx} + \\ 2u_3u_{0x} + 2u_1u_{2x} + 2u_3u_{0xxx} + 2u_1u_{2xxx} \\ + 2u_0u_{3xxx} + 2u_2u_{1xxx} \end{cases} = 0, & u_4(x,0) = 0, \\ \begin{cases} u_{5t} + 6u_{4x}u_{0xx} + 6u_{0x}u_{4xx} + 6u_{2x}u_{2xx} + \\ 6u_{3x}u_{1xx} + 6u_{1x}u_{3xx} + 2u_2u_{2x} + 2u_1u_{3x} \\ 2u_0u_{4x} + 2u_4u_{0x} + 2u_4u_{0xxx} + 2u_1u_{3xxx} \\ + 2u_3u_{1x} + 2u_3u_{1xxx} + 2u_0u_{4xxx} + 2u_2u_{2xxx} \end{cases} = 0, & u_5(x,0) = 0. \end{cases}$$

(3,16)

One can now try to obtain a solution for the above equation system in the form of:

$u_0(x,t) = x,$ (3.17.a)

$u_1(x,t) = x(-2t),$ (3.17.b)

$u_2(x,t) = x(4t^2),$ (3.17.c)

$u_3(x,t) = x(-8t^3),$ (3.17.d)

$u_4(x,t) = x(16t^4),$ (3.17.e)

$u_5(x,t) = x(-32t^5).$ (3.17.f)

Having $u_i, i = 0,1,...5$, the solution $\mathrm{u}(\mathrm{x},\mathrm{t})$ is therefore as:

$$u(x,t) = \sum_{i=0}^{5} u_i(x,t) = x\left[1 - 2t + (2t)^2 - (2t)^3 + (2t)^4 - (2t)^5\right]. \quad (3.18)$$

Higher iterations can make one obtain the exact solution of Eqs. (3.14, 3.15) in the form of $u(x,t) = \dfrac{x}{1+2t}$.

Example: Let us solve Eqs. (2.17, 2.18) through HPM; considering $n = 5$; thus, we will have the system equation as:

$$\begin{cases} u_{0t} = 0, & u_0(x,0) = x, \\ u_{1t} + u_0u_{0x} = 0, & u_1(x,0) = 0, \\ u_{2t} + u_1u_{0x} + u_0u_{1x} - u_{1xx} = 0, & u_2(x,0) = 0, \\ u_{3t} + u_0u_{2x} + u_2u_{0x} + u_1u_{1x} - u_{2xx} = 0, & u_3(x,0) = 0, \\ u_{4t} + u_3u_{0x} + u_0u_{3x} + u_2u_{1x} + u_1u_{2x} - u_{3xx} = 0, & u_4(x,t) = 0, \\ u_{5t} + u_3u_{1x} + u_1u_{3x} + u_4u_{0x} + u_0u_{4x} + u_2u_{2x} - u_{4xx} = 0 & u_5(x,t) = 0. \end{cases}$$

(3.19)

Trying to solve the system Eq. (3.19) results in:

$u_0(x,t) = x,$ (3.20.a)

$u_1(x,t) = x(-t),$ (3.20.b)

$u_2(x,t) = x(t^2),$ (3.20.c)

$u_3(x,t) = x(-t^3),$ (3.20.d)

$u_4(x,t) = x(t^4),$ (3.20.e)

$u_5(x,t) = x(-t^5),$ (3.20.f)

then,

$$u(x,t) = \sum_{i=0}^{5} u_i(x,t) = x(1 - t + t^2 - t^3 + t^4 - t^5). \quad (3.21)$$

Always by adding up the number of iterations one can attain the exact solution of Eqs. (2.17, 2.18) in the form $u(x,t) = \dfrac{x}{1+t}$.

Conclusion

The main goals of this study were the assessment of capability of the He's variational iteration method (VIM) and homotopy perturbation method (HPM) to solve the KdV type equations. The KdV, K(2,2), and Burgers equations that arise from many important and practical physical phenomenon were examined for rational solutions. Two above-mentioned methods were capable to solve this set of problems with successive rapidly convergent approximations without any restrictive assumptions or transformations causing changes in the physical properties of the problems. Also, adding up the number of iterations leads to the explicit solutions for the problems. Among two methods, VIM is very comprehensible as it reduces the size of calculations and also its iterations are direct and straightforward. HPM does not require small parameters in the equation so that the limitations of the conventional perturbation methods can be eliminated and thereby the calculations are simple and straight forward, though HPM can be more convenient.

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