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# Evaluation of benchmark information with Least-distance measurement model for non-convex frontiers and Petri Net

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## Abstract

Data Envelopment Analysis (DEA) is efficient analysis for similar decision making units (DMUs). It provides a ranking of DMUs relative to each other. DAE in addition to determining efficiency level, provides situations for removing inefficiency by using evaluated benchmark information. The Least-Distance Measurement model is proposed in order to shifting inefficiency to the nearest efficient unit which may not applied in real-world problems about convex technology (i.e., the convexity of the production probability function); thus, expanding DEA to non-convexity technology would be improved the potential of applying the Least-Distance Measurement model. This paper proposes extension of Least-Distance Measurement model in non-convexity technology and we will explore Petri Nets for modeling a least-distance measurement in non-convex space.

**Keywords**: Data Envelopment Analysis; Least distance; FDH; Benchmarking.

## Introduction

Data Envelopment Analysis (DEA), which first time has proposed by Charnes *et al*. (1978), is a non-parametric approach which evaluates performance of efficiency of various organizations in public and private sectors with multiple inputs and outputs. DEA is a mathematic programming approach which uses production frontiers for evaluating relative efficiency. If decision making unit (DMU) be depended to production frontiers, DMU is efficient; otherwise, DMU is inefficient. Each inefficient DMU try to be efficient and force a manager to find an efficient improvement for inefficient units of frontier. Some models in DEA are dealing with delineating inefficient DMU base on efficient frontier.

Free disposal hull (FDH) models, which first were formulated by Deprins *et al*. (1984), rely on this assumption that production of probabilities satisfy free disposability, and ensure the efficiency of evaluations are only affected by those performances which are actually observed (Jeong & Simar, 2006).

Each model in DEA has an imaginative procedure about inefficient DMUs based on efficiency of frontiers. However this research focuses on similarity or approximately of benchmarks, there are some important benchmarking attributes, such as evaluator preferences and organization landscape which should be considered. By the way, the Least-Distance Measurement benchmark information, by providing similar and easy attainable benchmarks in the lack of MPS or specific preferences, is continued to be meaningful and useful. Therefore, new Least-Distance measurement model is proposed by Beak and Lee (2009).

## Least-Distance measurement

Least-Distance measurement model in comprising with other models which dealing with supporting hyperplanes or pareto efficient faces, is computed by defining a strong efficient set. In fact, each inefficient DMU looks for improvement, situation for comprising and after an efficient DMU, achieving to the position which there is efficient DMU on efficient surfaces in it and preferably in this process, relatively. inefficient DMU was adjusted with pareto efficient frontier.

*Definition 1:* The Production Probability Set (pps) will be represented as

$$T=\{(x,y)\,|\,\text{y can produced by x}\}$$

*Definition 2:* The set of observations satisfying the pareto efficiency conditions are defined as a strongly efficient set, E, such that;

$$E=\left\{(x,y)\,\middle|\,\max\left(e^t s^+ + e^t s^-\right)=0 \right.$$
$$\left. s.t\ \left(s^+,s^-\right)=\left(x-X\lambda, Y\lambda-y\right),\ e^t\lambda=1,\ \lambda\ge 0\right\} \qquad (1)$$

Where

$$e^t=(1,1,...,1),$$

$$e^t s^+=\sum_{r=1}^{s} s_r^+,\quad e^t s^-=\sum_{i=1}^{m} s_i^-$$

The objective function of Least-Distance Measurement introduce the distance between the evaluated DMU, $\left(x^o,y^o\right)$, and the strongly efficient set $(E)$, into an efficiency measurement, and can be described as follows (Soleimani-damaneh & Mostefaee, 2009):

$$\theta=\max\left[1-\frac{1}{m+s}\left\{\sum_{i=1}^{m}\left(\frac{x_i-x_i^o}{R_i^-}\right)^2+\sum_{r=1}^{s}\left(\frac{y_r-y_r^o}{R_r^+}\right)^2\right\}^{1/2}\right]$$
$$s.t\quad (x,y)\in E \qquad (2)$$

where $R_i^- = \max_j \{x_{ij}\} - \min_j \{x_{ij}\}$

$R_r^+ = \max_j \{y_{rj}\} - \min_j \{y_{rj}\}$

m is the number of input variables, s is the number of output variables, $x_i^o$ i-th is the input of input vector $x^o$ and $y_r^o$ r-th is the output of output vector $y^o$.

Least-Distance Measurement model first describes strongly efficient set and then obtains distances between all combination of m+s components in set E of evaluated DMU (It's based on this fact that a point on a facet of the production frontier of m+s dimension can be showed by a linear combination of m+s members of set E) (Raty, 2002), and in regard to Least-Distance Measure, it obtains first efficient benchmark with nearest distance and the calculated measure of $\theta$.

### Free Disposal Hull (FDH) model

The FDH model first was formulated by Deprins, Simar and Tulens and then was developed and extended by Tulkens(1993). Primary motivation is obtaining this matter that evaluations efficiency just affected by those performances which are actually observed (Sueyoshi & Sekitani, 2007). The Production Probability Set of FDH can be described as follows;

$$PPS_{FDH} = \cup_{j=1}^{n} \left\{ (X, Y) \middle| X \geq x_j, Y \leq y_j \right\} = T_{FDH} \qquad (3)$$

The FDH input-oriented radial efficiency of $DMU_o(x_o, y_o)$, is obtained by solving the following model:

$$\begin{aligned} & \min \ \theta \\ & s.t \ \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io} \quad i=1,...,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} \quad r=1,...,s \\ & \sum_{j=1}^{n} \lambda_j = 1 \qquad (4) \\ & \lambda_j \in \{0,1\} \end{aligned}$$

Note that model (4) is a mixed integer programming.

### Petri Nets

Petri Nets can be applied to many areas such as the modeling of production lines, the design and analysis of workflows and business processes (Murata *et al.,* 1985; Zou & Zurawski, 1995). They can also simulate such processes with the help of a number of computer applications. Due to the fact that Petri nets can be described as a set of algebraic equations, they are considered as a powerful analysis tool. Petri nets can be used to check for the existence of deadlocks or starvation and analyze concurrency between processes precedence relations amongst events or the existence of appropriate synchronization (Zou & Zurawski, 1995).

They can also be used to measure the performance of the underlying system. Petri nets can be described both mathematically and graphically. For the sake of simplicity, this paper will not feature the mathematical description. Most of the Petri net diagrams presented in this document were made using Yasper (Yasper User Guide. Available at http://www.yasper.org/ (retrieved on 09/02/09)). To illustrate a difference in notation, another diagram was made using Woped. Graphically, Petri nets are described as a diagram with circles (places), bars or squares (transitions) and arrows (arcs) connecting them. Depending on the interpretation, the designer wishes to give them, places can represent conditions, input/output data or resources. Transitions can be interpreted as events, tasks or clauses, among others (Murata *et al., 1985).* Places can have multiple arcs from and to transitions and transitions can have multiple arcs from and to places. A transition can have arcs going back to its input places, symbolized in Fig.1 by a double arrow. A place can hold one or more tokens, symbolized by one or more dots. Depending on the interpretation given to places, a token can represent resources or whether a condition is true or false (Zou & Zurawski, 1995). In its basic incarnation, a transition is enabled and can fire when all the places that are connecting to it hold at least one token. When the transition fires, it removes (consumes) a token from all the incoming places and adds (produces) another token in all outgoing places.

### Least-distance measure in non-convex space

Consider the non-convex Production Probability Set (FDH). Similar to Least-Distance Measurement in convex space, that can't be contributed to the concept of combination of m+s components for efficiency of frontier in non-convex space efficient DMUs and deny to us favorable benchmarking the image of inefficient DMU, whereas eliminated for PPS in non-convex space. As it was mentioned, the aim is to find the nearest path for efficiency frontier in Least-Distance Measure model, therefore, it requires having the defining hyperplanes of PPS.

Consider a set of DMUs consisting of $DMU_1, DMU_2, ..., DMU_n$, where $DMU_j$, $(j \in J = \{1, 2, ..., n\})$, uses m positive inputs, $x_{ij}, (i = 1, 2, ..., m)$, to produce s positive outputs, $y_{rj}, (r = 1, 2, ..., s)$.

Let $DMU_j = (x_{1j}, x_{2j}, ..., x_{mj}, y_{1j}, y_{2j}, ..., y_{sj})$, $j = 1, ..., n$

### Contention 1

Concerning the dominate concept in Production Possibility Set of FDH, the pareto efficient frontier does not exist. There upon discussing the concept of Least-Distance Measure for efficiency frontier in non-convex space (FDH model). First we delineate the efficiency of frontier of inefficient DMU $DMU_o$ which consists of least feasible distances of efficient frontier of inefficient DMU (this process has performed by Euclidean norm). Concern the property of FDH frontier, which can be the only coefficient of vertex points that exist in combination and is used for solving FDH model with known vertex points and for designing inefficient DMU with various solved models which are located on one of vertex points. For obtaining the least distances of inefficient DMU for efficiency of frontier, compute the distances of $DMU_o$ to definded hyperplanes of reference DMU, and consider benchmarking of the efficient projection with the least measure. In Fig.1, the projection of inefficient DMU $DMU_o$ falls on $DMU_c$, which travers the references hyperplanes 3 times. therefore, it should be calculate the distance of inefficient DMU at each hyperplanes(It doesn't present their hyperplane).

Also, in this section similar to convex PPS, it is possible that the least distance of inefficient DMU be located on the region of PPS which isn't efficient$\left(s_1^-\right)$, therefore it has insufficient reference point for each DMU

*Fig. 1. The image of $DMU_o$ on non-convex frontier*

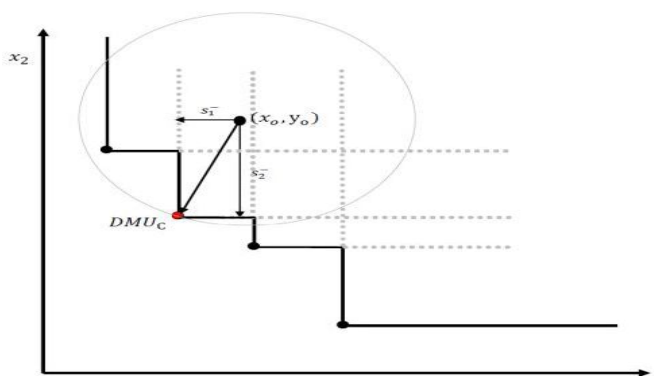
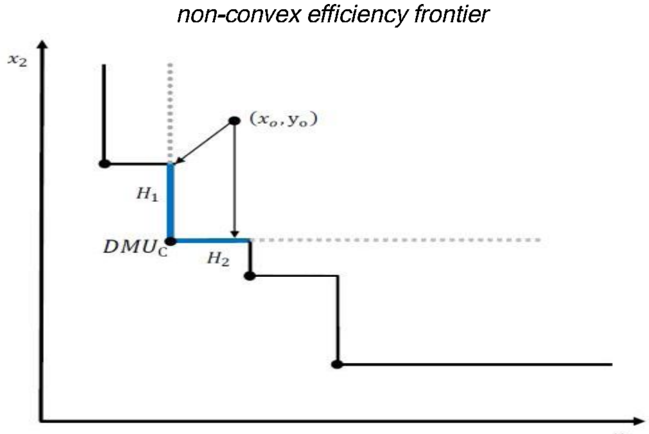



*Fig. 2. The nearest image of $DMU_o$ on non-convex efficiency frontier*




and their correspond hyperplanes and needs to the boundary of defined hyperplanes which form the efficient frontier of PPS, then we examine the least distance to efficient frontier in that region. Accordingly, in Fig 2 distances between two $H_1$ and $H_2$ segment is measured and the benchmarking on efficient frontier, i.e, the shortest path, is obtained. In fact, that perch on $H_1$ must moves as long as possess the least distance between hyperplanes in measure of $DMU_o$ and reference unit based on distinict definded hyperplanesThis purpose would be attained by moving into $\left(x+d_1\right)$ or $\left(y-d_2\right)$ which d designed for measuring the movements.

Operate on reminded efficient frontier needs to:

According to Fig.2, moving on $H_1$ and $H_2$ segments, should give approximately the scope of this hyperplanes which don't leave by moving along the hyperplanes from efficient frontier. It Will lead to this fact that each segments which these points are located on, merely dominated by reference unit ($DMU_c$) (Except those points of terminal segment which dominated by other vertex point). Taking into account of definition of dominated, relative d changes are:

1. Dominate reference unit where on $(x_c \le x+d_1, y_c \ge y-d_2)$.
2. Not Dominate the other vertex points $(x_k \ge x+d_1, y_k \le y-d_2$ which $k \ne c, k \in E$ ).

But in higher dimension of FDH frontier, whit out any possible determination priority of DMUs, condition 2 for those DMUs which have more measurement of inputs and less measurement of outputs for $DMU_c$ become practical. Further whereas it can be performed for all indexes except defined hyperplane indexes, implement it on one single defined hyperplane.

By taking into account Euclidean norm, we will obtain the nearest projection of inefficient DMU, and can determine the scope of defined hyperplane which form the frontier of PPS.

consider following algorithm which intuitively calculates the distance between position of single input and single output.

*1.1. The algorithm of Least-Distance measurement in non-convex space*

Assume that we have a set of n DMUs with m inputs and s outputs,

$$\left\{\left(X_j, Y_j\right)=\left(x_{1j}, x_{2j}, \ldots, x_{mj}, y_{1j}, y_{2j}, \ldots, y_{sj}\right)\right\},$$

where all inputs and outputs are positive.

*Step 1:* Separate DMUs with solved FDH model. Locate efficient DMUs in set E such that $|E|=L$ .

Perform following steps for inefficient DMU.

*Step 2:* Solve model (5) for each inefficient unit, which presents projection unit based on nearest vertex point of

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responsible model. (suppose that projection unit is located on $DMU_c$ and $c \in E$ ).

$$\begin{aligned} \min \ & \sum_{i=1}^{m}\left(x_i - x_i^o\right)^2 + \sum_{r=1}^{s}\left(y_r - y_r^o\right)^2 \\ s.t \ & x = X^E \lambda \\ & y = Y^E \lambda \\ & e^t \lambda = 1 \\ & \lambda \geq 0 \ , \ \lambda \in \{0,1\} \end{aligned} \qquad (5)$$

*step 3:* Find the passing hyperplanes of $DMU_c$ which include defined hyperplanes of PPS frontier.

The passing hyperplanes of $DMU_c : \begin{cases} x_i = x_{ic} \ , \ i=1,...,m \\ y_r = y_{rc} \ , \ r=1,...,s \end{cases}$,

Obtain the evaluated distance of DMU $DMU_o$ at hyperplanes.

If the hyperplanes are $x_k = x_{kc}$, they Should move in other dimension of inputs and all dimension of those outputs which remain on the frontier. This procedure presented as follows:

$$\begin{aligned} \min = & \sum_{\substack{i=1 \\ i \neq k}}^{m}\left(x_{io} - \left(x_i + d_{1i}\right)\right)^2 + \sum_{r=1}^{s}\left(y_{ro} - \left(y_r - d_{2r}\right)\right)^2 + \left(x_k - x_{ko}\right)^2 \\ s.t \ & \sum_{j \in E}\lambda_j x_{ij} + \lambda_{L+1}\left(x_i + d_{1i}\right) = \left(x_i + d_{1i}\right) \quad i=1,...,m \ , \ i \neq k \\ & \sum_{j \in E}\lambda_j y_{rj} + \lambda_{L+1}\left(y_r - d_{2r}\right) = \left(y_r - d_{2r}\right) \quad r=1,...,s \\ & \sum_{j \in E}\lambda_j x_{kj} + \lambda_{L+1} x_k = x_k \\ & x_k = x_{kc} \\ & x_i + d_{1i} \geq x_{ic} \quad i=1,...,m \ , \ i \neq k \\ & y_r - d_{2r} \leq y_{rc} \quad r=1,...,s \\ & x_i + d_{1i} \leq x_{it} \quad i=1,...,m \ , \ i \neq k \\ & y_r - d_{2r} \geq y_{rt} \quad r=1,...,s \\ & \lambda_j \in \{0,1\} \ , \ j=1,...,n \ , x_{ij} \geq 0 \ , y_{rj} \geq 0 \ , \\ & i=1,...,m \ , r=1,...,s \ , j=1,...,n \end{aligned} \qquad (6)$$

Where;

$$x_{it} = \left\{x_{ij} \,\middle|\, x_{ij} > x_{ic}, j = 1,...,n\right\}, \ y_{rt} = \left\{y_{rj} \,\middle|\, y_{rj} < y_{rc}, j = 1,...,n\right\}$$

$L$ is number of efficient set E members, which is reachable for three prime constraints of model about those points that are on the hyperplanes, other points are chosen in fourth constrain and those points that hyperplane, dominate reference unit on chosen hyperplane are define in fifth and sixth constraints, two remain constraints procedure movement on other dimensions of hyperplane such that other vertex point on hyperplane are dominate on chosen points (except one).

1. If the hyperplanes are $y_k = y_{kc}$, points Should move in other dimensions of output and all dimensions of input which remain on the frontier. This represented in following model:

$$\begin{aligned} \min = & \sum_{i=1}^{m}\left(x_{io} - \left(x_i + d_{1i}\right)\right)^2 + \sum_{\substack{r=1 \\ r \neq k}}^{s}\left(y_{ro} - \left(y_r - d_{2r}\right)\right)^2 + \left(y_k - y_{ko}\right)^2 \\ s.t \ & \sum_{j \in E}\lambda_j x_{ij} + \lambda_{L+1}\left(x_i + d_{1i}\right) = \left(x_i + d_{1i}\right) \quad i=1,...,m \\ & \sum_{j \in E}\lambda_j y_{rj} + \lambda_{L+1}\left(y_r - d_{2r}\right) = \left(y_r - d_{2r}\right) \quad r=1,...,s \ , \ r \neq k \\ & \sum_{j \in E}\lambda_j y_{kj} + \lambda_{L+1} y_k = y_k \\ & y_k = y_{kc} \\ & x_i + d_{1i} \geq x_{ic} \quad i=1,...,m \\ & y_r - d_{2r} \leq y_{rc} \quad r=1,...,s \ , \ r \neq k \\ & x_i + d_{1i} \leq x_{it} \quad i=1,...,m \\ & y_r - d_{2r} \geq y_{rt} \quad r=1,...,s \ , \ r \neq k \\ & \lambda_j \in \{0,1\} \ , \ j=1,...,n \ , x_{ij} \geq 0 \ , y_{rj} \geq 0 \ , i=1,...,m \ , r=1,...,s \ , j=1,...,n \end{aligned} \qquad (7)$$

This model hold all condition of model (5).

*Step 4:* Incrementaly Sort the optimal value of step 3, then select correspond initial point $(x, y)$ of chosen object value, and solve it for pair $(x, y)$ of FDH model, which is selected for benchmarking when there is efficient point, otherwise select the next point and continue this method until pair $(x, y)$ be efficient.

*Step 5:* $(x, y)$ is the nearest projection point from evaluated DMU to the efficient set E, so put it in model (2) and obtain the efficiency measurement of the Least-Distance Measurement. These steps are shown in Fig 3.

*4.2. Example*

Calculate the least distance measurement for that system which is shown in Fig 4. Data is given in Tab.1 The unit F which is accruing from FDH model, is the only inefficient DMU which has $E = \{A, B, C, D, E\}$

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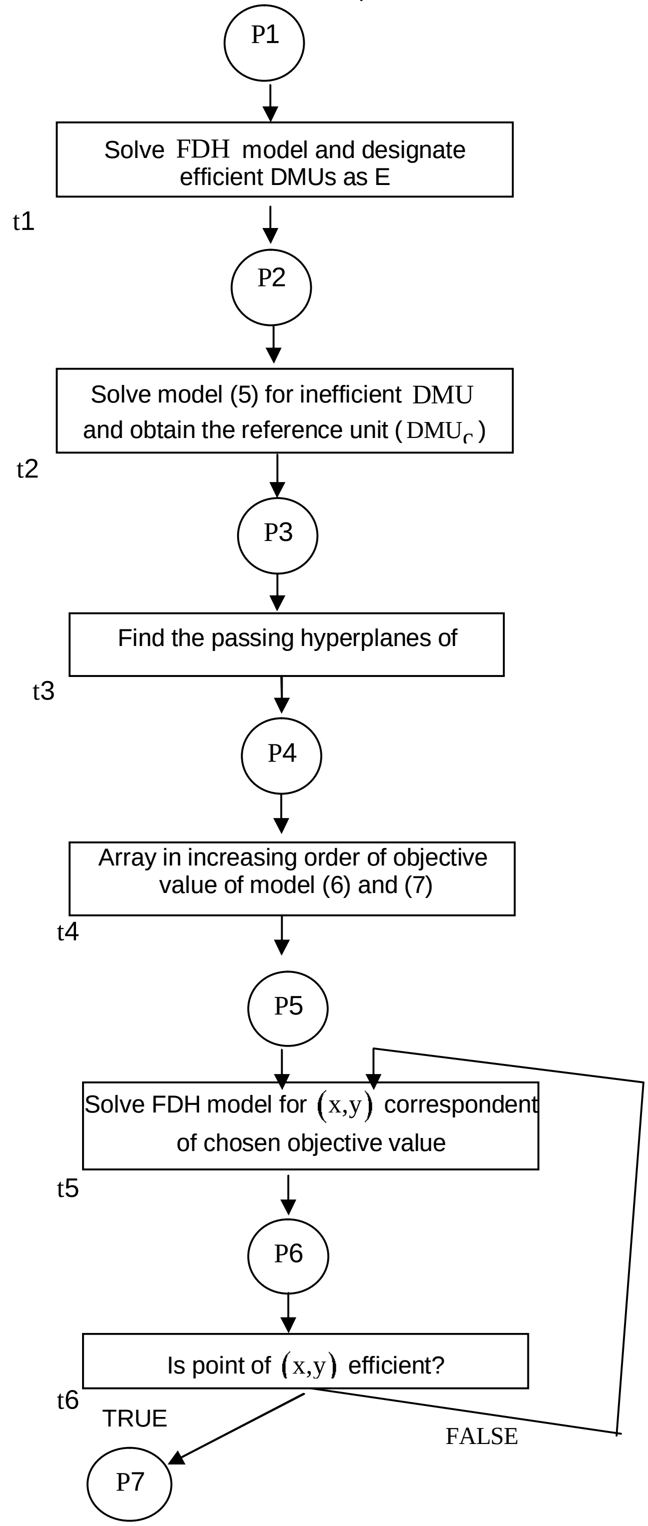
*Fig. 3. PETRI NET of the least-distance measure in non-convex space*




The reference unit for $DMU_F$ is $DMU_B$. therefore the passing hyperplanes of the $DMU_B$ is $x_1 = 3, x_2 = 6, y_1 = 1$ which lead to following results:

*Fig. 4. The FDH frontier in Example 1.*

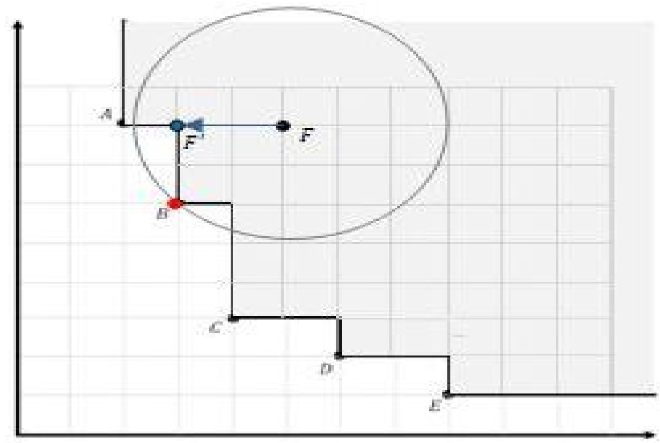
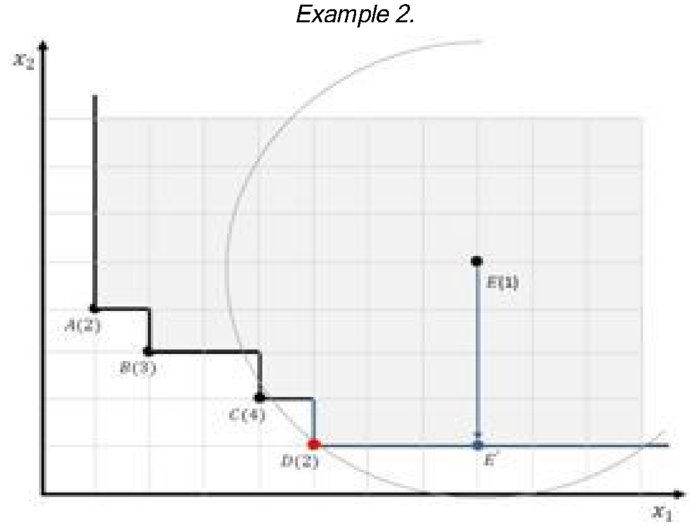



*Table 2. Numerical information for example 2*

| | A | B | C | D | E |
|---|---|---|---|---|---|
| $x_1$ | 1 | 2 | 4 | 5 | 9 |
| $x_2$ | 4 | 3 | 2 | 1 | 5 |
| $x_3$ | 2 | 3 | 4 | 2 | 1 |

The image on hyperplane $x_1 = 3$ is point $(3,8,1)$ with objective value 4, The image on hyperplane $x_2 = 6$ is point $(4,6,1)$ with objective value 5, The image on hyperplane $y_1 = 1$, point $(5,8,1)$ with objective value 0, which located on $x_1 = 3$ the initial efficiency point and therefore point $(3,8,1)$ is selected for benchmarking. This lead to objective value 0.96 with model 2.

*Fig. 5. The FDH frontier and measure of outputs in Example 2.*



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### *1.2.Another Example*

Calculate the least distance measurement for system as shown in Fig 5. Data is given in Table 2.

By solving the FDH model for units having $E = \{A, B, C, D\}$. Unit E is single inefficient DMU, which located on D by Euclidean norm. there are passing hyperplanes $x_1 = 5, x_2 = 1, y_1 = 2$. Then unit D is the solution from solving models (6) and (7).

The image on hyperplane $x_1 = 5$ is point $(5,2,1)$ with objective value 25, The image on hyperplane $x_2 = 1$ is point $(9,1,1)$ with objective value 16, The image on hyperplane $y_1 = 2$, point $(9,2,2)$ with objective value 10 so the selected benchmark is point $(9,1,1)$. This lead to objective value 0.67 with model(2).

## Conclusion

The efficiency value and benchmarking information can be obtained from DEA models, which provide situation for eliminating inefficiency and improving the state of inefficient unit for decision management, that is a process for finding unit with rely on efficiency frontier of the Least-Distance Measurement model. Further, in real-world problems, convex technology (i.e., the convexity of the production probability set) in regard of hypothesis of Least-Distance Measurement may not be applied for selecting the approximate unit based on non-convex efficiency frontier by passing hyperplanes of reference unit for benchmarking.

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