1. Introduction

There are many difficulties encountered in the application of perturbation techniques to the study of nonlinear problems. All classical perturbation techniques rely on the assumption of the small parameter. To overcome the limitations presented some approximate analytical methods to solve the nonlinear equations. There are many approximate analytical methods for solving the nonlinear equations, including the perturbation techniques, the homotopy methods, frequency-amplitude formulation, energy balance method, harmonic balance method, modified variational approach, and max-min method. In spite of the other perturbation techniques, the homotopy methods are applicable to strongly nonlinear systems. In this paper, we used this method for studying the vibration behavior of beams with damping nonlinearity. In this study, various finite element formulations to the large amplitude vibration of a hinged-hinged beam with immovable ends and presented an analytical formulation based on the Rayleigh-Ritz method. In this paper, we presented the Modified Homotopy Perturbation Method (MHPM) to study the large amplitude free vibration behavior of a pretensioned beam with clamped-clamped immovable ends. They modified He’s new perturbation technique and shown that their new presented method has higher accuracy than HPM and VIM. In this paper the Modified Homotopy Perturbation Method which can be used only for studying the free vibration analysis of beams is generalized to analyze the forced vibration cases. To this end, an Euler-Bernoulli clamped-clamped beam subjected to an external harmonic excitation which is rested on a Pasternak foundation is assumed. Applying the Von-
Generalizing Modified Homotopy Perturbation Method to Study the Large Amplitude Vibration of Beams Subjected to an External Harmonic Excitation

Karman nonlinear strain-displacement relation and the Newton's second law and by implementing the Galerkin's Method, the nonlinear equation of motion is derived. To solve this nonhomogeneous strongly nonlinear equation, the MHPM which is already presented by is generalized. For validating the results of MHPM, some experimental tests are carried out. Moreover, the time response of the first and second order generalized MHPM follows accurately the time response obtained by the Runge-Kutta Method.

2. Equation of Motion

A schematic of Euler-Bernoulli beam with a length of \( L \), cross-sectional area of \( A \), density of \( \rho \), area moment of inertia of \( I \) and the elasticity modulus of \( E \), resting on a Pasternak foundation is shown in Figure 1. Considering an element of the beam as Figure 2 and using the Newton's second law, one can obtain the equation of motion. \( f \) represents the reaction force of the foundation.

\[
f = k_L w + k_{NL} w^* \tag{1}
\]

Where \( k_L \) and \( k_{NL} \) are linear and nonlinear foundation stiffness, respectively.

The equilibrium of moments around point \( O \) is written

\[
\sum M_O = 0 \rightarrow -\nu dx + M - \left( M + \frac{\partial M}{\partial x} dx \right) - f \left( \frac{dx}{2} \right) = 0 \Rightarrow \nu = -\frac{\partial M}{\partial x} \tag{2}
\]

The strain-displacement relations for a beam undergoing large deflections are as:

\[
\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w^2}{\partial x^2} \right), \quad \kappa_x = -\frac{\partial^2 w}{\partial x^2} \tag{3}
\]

Where \( u \) is the longitudinal displacement, \( w \) is the lateral displacement, and \( x \) is the longitudinal coordinate. The bending moment will be as:

\[
M = -EI\kappa_x = EI \frac{\partial^2 w}{\partial x^2} \tag{4}
\]

The value of inertial force is \( \rho A \frac{\partial^2 w}{\partial t^2} dx \) and its direction will be downward.

The sum of all forces in the \( y \)-direction or vertical is as:

\[
\sum F_y = -\nu + \left( \nu + \frac{\partial V}{\partial x} dx \right) - P \sin \theta + \left( P + \frac{\partial P}{\partial x} dx \right)
\]

With some simplifications, the Equation (5) is rewritten as:

\[
\frac{\partial V}{\partial x} dx + P \frac{\partial \theta}{\partial x} dx - f dx = \frac{\partial^2 M}{\partial x^2} dx + P \frac{\partial^2 W}{\partial x^2} dx
\]

\[
- k_L w dx - k_{NL} w^* dx = \rho A \frac{\partial^2 w}{\partial t^2} dx \tag{6}
\]

The value of force \( P \) is assumed as:

\[
P = P_0 + P_1 \tag{7}
\]

Where \( P_0 \) is the initial pretension force in the beam and \( P_1 \) is the initial force due to mid-plane stretching and its value is:

\[
P_1 = \varepsilon_x E A \tag{8}
\]

Integrating from strain relation in Equations (3) and assuming immovable boundary conditions, one obtains:

\[
u(t) - u(0, t) = \int_0^l \varepsilon_x dx - \frac{1}{2} \int_0^l \left( \frac{\partial w}{\partial x} \right)^2 dx = 0 \tag{9}
\]

Consequently from Equations (9), we have:

\[
\varepsilon_x = \frac{1}{2l} \int_0^l \left( \frac{\partial w}{\partial x} \right)^2 dx \tag{10}
\]

From Equations (7), (8) and (10), one obtains:

\[
P = P_0 + \frac{EA}{2l} \int_0^l \left( \frac{\partial w}{\partial x} \right)^2 dx \tag{11}
\]

If a concentrated vertical force, \( F \), is applied on the beam at distance \( x_b \) from the left side of the beam, and using Equations (4), (6) and (11) yields:

\[
E I \frac{\partial^4 w}{\partial x^4} - P_0 \frac{\partial^2 W}{\partial x^2} - \frac{EA}{2l} \left( \int_0^l \left( \frac{\partial w}{\partial x} \right)^2 dx \right) \frac{\partial^2 W}{\partial x^2} dx + \rho A \frac{\partial^2 w}{\partial t^2} dx
\]

\[
+ k_L w + k_{NL} w^* = F \delta(x - x_b) \cos \Omega t \tag{12}
\]

Figure 1. A clamped-clamped beam subjected to an external harmonic excitation and rested on Pasternak foundation.
3. Non-Dimensionalization of Equation of Motion

It is common and efficient to work with the dimensionless quantities. So, the dimensionless quantities are defined as:

\[ \tilde{t} = \omega_i t, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{w} = \frac{w}{w_{\text{max}}} \]  \hspace{1cm} (13)

Where \( \tilde{x} \) is the number of excited mode and \( \omega_i \) is the corresponding linear natural frequency and it is defined as:

\[ \omega_i = \beta_i L \sqrt{EI} \rho A L^4 \]  \hspace{1cm} (14)

\( \beta_i L \) is the eigenvalue of the beam with clamped-clamped boundary conditions. Substitution of Equation (13) and Equation (14) in to Equation (12) yields:

\[ \frac{\partial^4 \tilde{w}}{\partial \tilde{x}^4} - \frac{P_0}{EA} \left( \frac{L}{r} \right)^2 \frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2} \right) = \frac{1}{2} \left( \frac{L}{r} \right)^2 \left( \int_0^1 \left( \frac{\partial \tilde{w}}{\partial \tilde{x}} \right) \, d\tilde{x} \right) \left( \frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2} \right) + \beta_i^4 L^4 \frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2} \]  \hspace{1cm} (15)

Applying Galerkin technique to Equation (16) results in a second order nonlinear ordinary differential equation as:

\[ \left( \beta_i L \right)^4 \left( \int_0^1 \varphi' \, d\tilde{x} \right) \tilde{q} + \left( \int_0^1 \varphi' \varphi'' \, d\tilde{x} \right) q + \frac{P_0}{EA} \left( \frac{L}{r} \right)^2 \left( \int_0^1 \varphi' \, d\tilde{x} \right) \tilde{q} = \left( \frac{1}{2} \left( \frac{L}{r} \right)^2 \left( \int_0^1 \varphi \varphi' \, d\tilde{x} \right) \left( \int_0^1 \varphi \varphi'' \, d\tilde{x} \right) + \frac{k_{NL} L^4}{EA} \left( \frac{L}{r} \right)^2 \left( \int_0^1 \varphi' \, d\tilde{x} \right) \right) \tilde{q}^3 = \frac{FL^3}{EI} \varphi(0.5) \cos \Omega \tilde{t} \]  \hspace{1cm} (16)

Equation (17) can be rewritten as:

\[ \tilde{q} + \omega_i^2 \tilde{q} + \beta_i q^3 = \tilde{F} \cos \tilde{\Phi} \]  \hspace{1cm} (18)

Where:

\[ \omega_i^2 = \frac{f_4 EI - f_2 P_0 L^2 + f_1 k_{NL} L^4}{f_1 EI \beta_i^4 L^4} \]  \hspace{1cm} (19)

\[ \beta_i = \frac{-EA f_2 f_5 + 2f_5 k_{NL} L^4}{2f_1 EI \beta_i^4 L^4} \]  \hspace{1cm} (20)

The initial conditions of the beam are considered as:

\[ q(0) = 0, \quad \dot{q}(0) = 0 \]  \hspace{1cm} (21)

From Equation (18) and Equation (21), to take account of the external loading and the initial conditions, the response of the system is assumed as:

\[ q(\tilde{t}) = \tilde{q} \cos \tilde{\Phi} - \cos \tilde{\theta} \]  \hspace{1cm} (22)

In which \( \tilde{q} \) is the amplitude of vibration and \( \tilde{\theta} \) is the
correction frequency. These both unknown coefficients will be obtained using MHPM. The initial approximation is assumed as:

\[ q_0(t) = \bar{Y} \cos \Omega t \]  

(23)

Based on the MHPM, the terms \( q \) and \( \Omega^2 \) are considered as below:

\[ q = q_0 + \beta_1 q_1 + \beta^2 q_2 + \ldots \]  

(24)

\[ \Omega^2 = \omega_0^2 + \beta \omega_1 + \beta^2 \omega_2 + \ldots \]  

(25)

Considering \( \bar{F} = \bar{F} \), and substituting the Equation (24) and Equation (25) in to Equation (18) yields:

\[ \beta_1: \frac{\bar{d}^2 q_1}{\bar{d}t^2} + \omega_0^2 q_1 = \omega_1 q_0 - q_0^3 + \bar{F} \cos \Omega t \]  

(26)

\[ \beta_2: \frac{\bar{d}^2 q_2}{\bar{d}t^2} + \omega_0^2 q_2 = \omega_1 q_1 + \omega_2 q_0 - 3q_0^2 q_1 \]  

(27)

By substituting Equation (23) in to Equation (26) we have:

\[ \frac{\bar{d}^2 q_1}{\bar{d}t^2} + \omega_0^2 q_1 = \left( \omega_1 \bar{Y} - \frac{3}{4} \bar{Y}^3 + \bar{F} \right) \cos \Omega t - \frac{1}{4} \bar{Y}^3 \cos 3\Omega t \]  

\[ = 0 \]  

(28)

In order to avoid the secular term, the coefficient of \( \cos \Omega t \) will be equated to zero, so:

\[ \omega_1 \bar{Y} - \frac{3}{4} \bar{Y}^3 + \bar{F} = 0 \]  

(29)

Then:

\[ \omega_1 = \frac{3}{4} \bar{Y}^2 - \frac{\bar{F}}{\bar{Y}} \]  

(30)

Using Equation (25), assuming only the first order approximate solution and neglecting \( O(\beta^2) \), we obtain:

\[ \omega_1 = \frac{\Omega^2 - \omega_0^2}{\beta} \]  

(31)

From Equation (30) and Equation (31), one reaches:

\[ \left( \frac{\Omega^2 - \omega_0^2}{\beta} \right) \bar{Y} - \frac{3}{4} \bar{Y}^3 + \bar{F} = 0 \]  

(32)

Then:

\[ \frac{3}{4} \beta \bar{Y}^3 - \left( \Omega^2 - \omega_0^2 \right) \bar{Y} - \bar{F} = 0 \]  

(33)

Solving the equation, the value of \( \bar{Y} \) will be defined for the first approximation. In addition, the frequency response can be determined by defining the value of \( \bar{Y} \) for varying excitation frequencies. It is also worth mentioning that the Equation (18) has cubic nonlinearity; therefore, super harmonic resonance at \( \omega_r = \frac{3}{2} \Omega \) with sub harmonic resonance at \( \omega_r = 3\Omega \) will be occurred. Next, in order to determine the correction frequency, \( \bar{\sigma} \), a same procedure as one demonstrated for obtaining the nonlinear resonance frequency is used. To this aim, the initial conditions are assumed to be the same as one considered for free vibration analysis with \( q(0) = a_0 = \bar{Y} \) and \( \dot{q}(0) = 0 \), in which \( \bar{Y} \) is the amplitude of vibration obtained from Equation (33) previously. Using MHPM and considering the first order approximation, one can obtain:

\[ \bar{\sigma} = \sqrt{\omega_0^2 + \frac{3}{4} \beta \bar{Y}^2} \]  

(34)

Therefore, knowing \( \bar{Y} \) and \( \bar{\sigma} \), the time response of the beam under the harmonic load is defined from Equation (22) for the first order approximation. To obtain the second order approximate solution, \( q_1 \) will be obtained by solving the Equation (28):

\[ q_1(t) = \frac{\bar{Y}^3}{4 \left( 9 \Omega^2 - \omega_0^2 \right)} \left( \cos 3\Omega t - \cos \Omega t \right) \]  

(35)

Substituting Equation (23) and Equation (35) into Equation (27) yields:

\[ \frac{\bar{d}^2 q_2}{\bar{d}t^2} + \omega_0^2 q_2 = \frac{3}{4} \bar{Y}^2 - \frac{\bar{F} \bar{Y}^3}{4 \left( 9 \Omega^2 - \omega_0^2 \right)} \left( \cos 3\Omega t - \cos \Omega t \right) \]  

(36)

Then:

\[ \frac{\bar{d}^2 q_2}{\bar{d}t^2} + \omega_0^2 q_2 = \left( \frac{3\bar{Y}^2}{16 \left( 9 \Omega^2 - \omega_0^2 \right)} + \frac{\bar{F} \bar{Y}^3}{4 \left( 9 \Omega^2 - \omega_0^2 \right)} \right) \cos 3\Omega t - \left( \frac{3\bar{Y}^2}{16 \left( 9 \Omega^2 - \omega_0^2 \right)} \right) \cos \Omega t \]  

(37)

To avoid the secular term, the coefficient of \( \cos \Omega t \) will be equated to zero, therefore:
Then:

\[
\omega_2 = -\frac{3\dot{\gamma}^4}{16\left(9\Omega_0^2 - \omega_0^2\right)} - \frac{\dot{\gamma}^5}{4\left(9\Omega_0^2 - \omega_0^2\right)}
\]  
(39)

Substituting Equation (30) and Equation (39) in to the Equation (25), assuming only the second order approximate solution and neglecting \(O(\dot{\gamma}^2)\), we obtain:

\[
\hat{\Omega}^2 = \omega_0^2 + \frac{3\dot{\gamma}^2}{4\hat{\beta}^2} - \frac{\ddot{\gamma}^5}{16\left(9\Omega_0^2 - \omega_0^2\right)} - \frac{3\dot{\gamma}^4}{4\left(9\Omega_0^2 - \omega_0^2\right)} - \frac{\dot{\gamma}^5}{4\left(9\Omega_0^2 - \omega_0^2\right)}
\]  
(40)

After some mathematical manipulations, we reach:

\[
3\hat{\beta}^2\dot{\gamma}^5 + \left(-16\beta^2\dot{\gamma}^2 + 12\beta\dot{\gamma}\right)\ddot{\gamma}^2 + 4\dot{\gamma}^2\ddot{\gamma}^5 = 0
\]  
(41)

As mentioned before, the correction frequency, \(\hat{\beta}\), is the same as the nonlinear resonance frequency for the free vibration system with initial conditions \(q(0) = a_0 = \ddot{\gamma}, \quad q(0) = 0\). Where the value of \(\ddot{\gamma}\) can be calculated from Equation (41). Using MHPM and considering the second order approximation, one can obtain:

\[
\hat{\beta} = \frac{1}{4}\sqrt{8\omega_0^2 + 6\ddot{\gamma}^2 + 9\ddot{\gamma}^2 + 30\dot{\gamma}^2}
\]  
(42)

4. Validating the Results of MHPM with the Experiments

To validate the MHPM results, some experimental tests were carried out on the clamped-clamped steel beam with the given characteristics in Table 1. The beam is subjected to various initial displacements at its mid-point and the acceleration response of the beam was captured using a 4507 B&K accelerometer and a 3109 B&K signal analyzer. The test setup is shown in Figure 3. In Table 2, the measured linear and nonlinear natural frequencies of the beam for various values of initial displacements are compared with the nonlinear natural frequencies obtained by the MHPM.

As it can be seen, for various values of vibration amplitudes, there is a good agreement between the results obtained from the MHPM and the experiments. Figure 4 shows the variation of the nonlinear fundamental frequency of the beam against the maximum displacement at its mid-span. As it is seen, as the initial displacement at the beam mid-span increases, the nonlinear fundamental frequency rises. Moreover, this figure shows that the nonlinear fundamental frequencies obtained by the MHPM closely match with the corresponding experiments and the relative error is lower than 0.76%.

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<td>149.7</td>
<td>136.0</td>
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</tr>
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</table>

Table 1. The characteristics of the beam

| 3.9 | b (mm) |
| 6.4 | h (mm) |
| 485 | L (mm) |
| 7860 | \(\rho \text{ (kg/m}^3\text{)}\) |
| 190 | E (GPa) |
| 0.3 | \(\nu\) |

Table 2. Frequencies of the beam obtained from MHPM method and experiments
show that the second order of generalized MHPM follows the RK45 more accurate than the first order of generalized MHPM. Consequently, using this new presented method, the strongly nonlinear vibrational response of the beam under an external harmonic excitation is accurately derived.

5. Validating the Results of Generalized MHPM with Numerical Solution

After validating the MHPM results with the experiments, here we decide to validate the generalized MHPM results with the numerical solution. Figure 5 indicates the comparison between the time responses of forced vibration of the beam obtained by the first order of generalized MHPM with those obtained by the RK45. Moreover, in Figure 6, the time responses of forced vibration of the beam obtained by the second order of generalized MHPM with those obtained by the RK45 are compared. As these figures reveal, the results are in a very good agreement with each other. In addition, these figures

Figure 3. Set up for the experimental tests.

Figure 4. Variation of nonlinear frequency versus displacement.

Figure 5. Comparing the responses obtained by the first order MHPM and the numerical method for the clamped-clamped beam under external harmonic excitation.

Figure 6. Comparing the responses obtained by the second order MHPM and the numerical method for the clamped-clamped.

6. Conclusion

In this study, large amplitude vibration behavior of an Euler-Bernoulli beam subjected to an external harmonic
excitation which is rested on a Pasternak foundation is investigated. It is assumed that the beam vibrates only in a single mode. Employing the Von-Karman strain-displacement relations, using the Newton’s second law and then implementing the Galerkin’s method, the nonlinear ODE is derived. The coefficient of the nonlinear term would be very larger than unity; therefore, the traditional perturbation methods which are based on the small coefficient of the nonlinear term lead to an invalid solution. To solve the obtained strongly nonlinear non-homogeneous equation, the Modified Homotopy Perturbation Method (MHPM) is generalized. Also, to validate the results of MHPM, an experimental test was carried out. The results of analytical and experimental investigation were in a very good agreement. Moreover, the time response of the first and second order generalized MHPM follows accurately the time response obtained by Runge-Kutta solution.

6. References

19. A theoretical and experimental investigation on large amplitude free vibration behavior of a pre-tensioned beam with clamped-clamped ends using Modified Homotopy Perturbation Method. 2015. Available from: http://pic.sagepub.com/content/early/2015/04/14/0954406215580663.abstract