The existence of anti-periodic solution for a class of cellular neural networks

Zhouhong Li, Chenxi Yang and Kaihong Zhao
Department of Mathematics, Yuxi Normal University, Yuxi, Yunnan, China-653 100
zhouhli@yeah.net

Abstract

In this paper, we use the Lyapunov function to establish new results on the existence and uniqueness of anti-periodic solutions for a class of cellular neural networks with time-varying delays and continuously distributed delays of

\[ x_i(t) = -d_i(t)h_i(t, x_i(t)) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t-\tau_j(t))) + \sum_{j=1}^{n} b_{ij}(t) \times \int_{0}^{\infty} K_{ij}(s) g_j(x_j(t-s))ds + I_i(t), i = 1, 2, \ldots, n. \]

Moreover, we also present an example to illustrate the feasibility and effectiveness of our results.

Keywords: Cellular neural networks, distributed delays, anti-periodic solution, exponential stability, delays.

1. Introduction

Cellular neural networks (CNNs) have been widely studied both in theory and applications (Roska & Vandesande, 1995; Chua & Yang, 1988; Li, 2004; Cheng et al., 2006; Chen 2002; Cao & Wang, 2005; Mohamad, 2007). They have been successfully applied to signal processing, pattern recognition, optimization and associative memories, especially in image processing and solving nonlinear algebraic equations. Recently, Peng & Huang (2009) and Shao (2008) have studied the existence and stability of anti-periodic solution of the following shunting inhibitory cellular neural networks with time-varying delays. In this paper, we consider the following shunting inhibitory cellular neural networks with delays of continuous function on \( \tau = -\infty \), \( \cdot \) and \( \cdot \). Moreover, it will be assumed that \( I_i(t), i = 1, 2, \ldots, n \) are continuous function on \( R \), in which \( n \) corresponds to the number of units in a neural network, \( x_i(t) \) corresponds to the state of the \( i \)th unit at the time \( t \), \( \tau_j(t) \geq 0 \) corresponds to the transmission delay of the \( i \)th unit along the axon of the \( j \)th unit at the time \( t \), and \( I_i(t) \) denote the external inputs at time \( t \), \( f_j(\cdot) \) and \( g_j(\cdot) \) are activation functions of signal transmission. Our main purpose of this paper is by constructing Lyapunov functions to investigate the stability and existence of anti-periodic solutions of (1.1) and to give the conditions for the existence and exponential stability of the anti-periodic solutions for system (1.1), which are new and complement previously known results. Moreover, an example is also provided to illustrate the effectiveness of our results.

2. Notations and preliminaries

For convenience, we consider model (1.1) under some following assumptions.

Let \( u(t) : R \to R \) be continuous in \( t \). \( u(t) \) is said to be \( T \)-anti-periodic on \( R \), if

\[ u(t + T) = -u(t) \quad \text{for all} \quad t \in R. \]

If a system is \( T \)-anti-periodic \( (x(t + T) = -x(t)) \), then it is \( 2T \)-periodic \( (x(t + 2T) = -x(t)) \).

Throughout this paper, for \( i, j = 1, 2, n \) and for all \( t, u \in R \), it will be assumed that

\[ T > 0, d_i(t) > 0, (t + T)h_i(t + T, u) = -d_i(t)h_i(t, -u). \] (2.1)

\[ a_j(t + T)f_j(u) = -a_j(t)f_j(-u), h_i(t + T)g_j(u) = -b_j(t)g_j(u). \] (2.2)

\[ I_i(t + T) = -I_i(t), \tau_j(t + T) = \tau_j(t). \] (2.3)

Then, we suppose that there exits constants \( \overline{I}_i, \overline{a}_j, \overline{b}_j \), and \( \tau \) such that

\[ 0 < \overline{I}_i = \sup_{t \in R}|I_i(t)|, \quad \overline{a}_j = \sup_{t \in R}|a_j(t)|, \quad \overline{b}_j = \sup_{t \in R}|b_j(t)|, \quad \tau = \max_{1 \leq i,j \leq n} \left\{ \max_{t \in [0,T]} \tau_{ij} \right\}. \]
We also assume that the following conditions hold. 
(T0) for \( i \in \{1, 2, ..., n\} \), \( h_i(t, u) : R \rightarrow R^2 \) are continuous function, and there exist nonnegative constant \( h_i > 0 \) such that 
\[
h_i(t, 0) = 0, h_i \left| u - v \right| \leq \text{sgn}(u - v)(h_i(t, u) - h_i(t, v)),
\]
for all \( u, v \in R \).
(T1) for each \( j \in \{1, 2, ..., n\} \), there exist nonnegative constants \( F_j, G_j \) such that 
\[
f_j(0) = 0, f_j(u) - f_j(v) \leq F_j \left| u - v \right|, \quad g_j(0) = 0, g_j(u) - g_j(v) \leq G_j \left| u - v \right|,
\]
for all \( u, v \in R \).
(T2) for \( i, j \in \{1, 2, ..., n\} \), take delay kernel 
\[
K_{ij} \in C(R^+, R) \text{ are continuous integrable, and satisfies}
\]
\[
\int_0^\infty |K_{ij}(s)| ds \leq k_{ij}.
\]
(T3) there exist constants \( \beta > 0, \lambda > 0 \) and \( \zeta_i > 0, i = 1, 2, ..., n \) such that for all \( t > 0 \), there holds 
\[
-d_i(t)h_i \zeta_i + \sum_{j=1}^n a_{ij}F_j \zeta_j + \sum_{j=1}^n b_{ij}G_j \zeta_j k_{ij} < -\beta < 0,
\]
for \( i, j = 1, 2, ..., n \).

For convenience, we introduce some notations. We will use \( x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in R^n \) to denote a column vector in which they symbol \((i)^T\) denotes the transpose of a vector. We let \( |x| \) denote absolute-value vector given by \( |x| = (|x_1|, |x_2|, ..., |x_n|)^T \), and define 
\[
\|x\| = \max_{1 \leq i \leq n} |x_i|.
\]

The initial conditions associated with system (1.1) are of the form 
\[
x_i(s) = \phi_i(s), s \in (-\infty, 0], i = 1, 2, ..., n,
\]
where \( \phi_i(\cdot) \) denote a real-value bounded continuous function defined on \( (-\infty, 0] \). Denote 
\[
R^+ = [0, \infty), R_+ = (-\infty, 0].
\]

Definition 2.1. Let \( x^*(t) = (x_1^*(t), x_2^*(t), ..., x_n^*(t))^T \) be an anti-periodic solution of system (1.1) with initial value \( \phi^*(t) = (\phi_1^*(t), \phi_2^*(t), ..., \phi_n^*(t))^T \). If there exist constants \( \lambda > 0 \) and \( M > 1 \) such that for every solution \( Z(t) = (x_1(t), x_2(t), ..., x_n(t))^T \) of system (1.1) with any initial value \( \phi(t) = (\phi_1(t), \phi_2(t), ..., \phi_n(t))^T \), 
\[
\|x_i(t) - x_i^*(t)\| \leq M \|\phi - \phi^*\| e^{-\lambda t}, \quad \forall t > 0, i = 1, 2, ..., n,
\]
where 
\[
\|\phi - \phi^*\| = \sup_{-\infty < s \leq 0} \max_{1 \leq i \leq n} \|\phi_i(s) - \phi_i^*(s)\|,
\]
then \( x^*(t) \) is said to be globally exponentially stable.

The following lemmas will be used to prove our main results in Section 3.

Lemma 2.1. [15] Let (T0), (T1), (T2) and (T3) hold, suppose that \( \tilde{x}(t) = (\tilde{x}_1(t), \tilde{x}_2(t), ..., \tilde{x}_n(t))^T \) is a solution of system (1.1) with initial conditions 
\[
\tilde{x}_i(s) = \tilde{\phi}_i(s), \quad |\tilde{\phi}_i(s)| < \zeta_i^T, s \in R, i = 1, 2, ..., n.
\]
Then
\[
|\tilde{x}_i(t)| < \zeta_i^T \quad \text{for all} \quad t > 0, i = 1, 2, ..., n.
\]

Proof. Assume, by way of contradiction, that (2.2) does not hold. Then, there must exist 
\( i \in \{1, 2, ..., n\} \) and \( t_0 > 0 \) such that 
\[
|\tilde{x}_i(t_0)| = \zeta_i^T, \quad \text{and} \quad |\tilde{\phi}_i(s)| < \zeta_i^T \quad \text{for all} \quad t \in (-\infty, t_0], j = 1, 2, ..., n.
\]
Calculating the upper left derivative \( |\tilde{x}_i(t)| \), together with (T0), (T1), (T2) and (T3) implies that 
\[
0 \leq D^+ (|\tilde{x}_i(t)|) \]
\[
\leq \left[-d_i(t_0)h_i \zeta_i + \sum_{j=1}^n a_{ij}F_j \zeta_j + \sum_{j=1}^n b_{ij}G_j \zeta_j k_{ij} \right] + I_i(t_0) |\tilde{\phi}_i(s)| \zeta_i^T < \frac{\beta}{\lambda} |\tilde{x}_i(t_)0| + I_i(t_0).
\]
Which is a contradiction and implies that (2.6) holds, the proof of Lemma 2.1 is now completed.

Remark 2.1. In view of the roundedness of this solution, from the theory of functional differential equations by Hale (1977), it follows that \( \tilde{x}(t) \) on \( R^+ \).
\( \varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T \). Assume also the following condition is satisfied.

(T4) there exist constants \( t > 0 \), \( \lambda > 0 \) and \( \xi_j > 0 \), \( i = 1, 2, \ldots, n \) such that for all \( t > 0 \), there holds

\[
(\lambda - d_i(t) h_i(t)) \xi_j + \sum_{j=1}^{n} b_j(t) \int \mathcal{K}_j(s) g_j(y_j(t-s) + x_j(t-s)) \, ds \leq 0,
\]

Then there exist constants \( M_\varphi > 1 \) such that

\[
\left| x_i(t) - x_i^* (t) \right| \leq M_\varphi \left\| \varphi - \varphi^* \right\| e^{-\lambda t}, \quad \text{for all} \quad t > 0, \ i = 1, 2, \ldots, n.
\]

**Proof.**
Let \( y(t) = \{ y_j(t) \} = x_j(t) - x_j^* (t) = x(t) - x^* (t) \). Then

\[
y_j(t) = \sum_{j=1}^{n} b_j(t) \int \mathcal{K}_j(s) g_j(y_j(t-s) + x_j(t-s)) \, ds + \int \mathcal{K}_j(s) g_j(y_j(t-s) + x_j^*(t-s)) \, ds
\]

where \( i = 1, 2, \ldots, n \).

We consider the Lyapunov functional

\[
V_i(t) = \left| y_j(t) \right| e^{\lambda t}, \ i = 1, 2, \ldots, n.
\]

Calculating the left upper derivative of \( V_i(t) \) along the solution \( y(t) = y_j(t) \) of system (2.4) with the initial value \( \varphi = \varphi^* \), form (2.7), we have

\[
D^+(V_i(t)) \leq \lambda \left| y_j(t) \right| e^{\lambda t} + \sum_{j=1}^{n} b_j(t) \int \mathcal{K}_j(s) g_j(y_j(t-s) + x_j(t-s)) \, ds + \int \mathcal{K}_j(s) g_j(y_j(t-s) + x_j^*(t-s)) \, ds
\]

\[
\leq \lambda \left| y_j(t) \right| e^{\lambda t} + \sum_{j=1}^{n} b_j(t) \int \mathcal{K}_j(s) g_j(y_j(t-s) + x_j^*(t-s)) \, ds + \sum_{j=1}^{n} b_j(t) \int \mathcal{K}_j(s) g_j(y_j(t-s) + x_j^*(t-s)) \, ds
\]

Thus,

\[
\left| x_i(t) - x_i^* (t) \right| \leq M_\varphi \left\| \varphi - \varphi^* \right\| e^{-\lambda t}, \quad \text{for all} \quad t > 0.
\]

Where \( i = 1, 2, \ldots, n \). Let \( m > 1 \) denote an arbitrary real number such that

\[
m_i \xi_j > \left\| \varphi - \varphi^* \right\| = \sup_{-\infty < s < 0} \max_{1 \leq i \leq n} \varphi_j(s) - \varphi_j^*(s), \quad i = 1, 2, \ldots, n.
\]

It following from (2.8) that

\[
V_i(t) = \left| y_j(t) \right| e^{\lambda t} < m_i \xi_j, \quad \text{for all} \quad t > 0, \ i = 1, 2, \ldots, n.
\]

We claim that

\[
V_i(t) = \left| y_j(t) \right| e^{\lambda t} < m_i \xi_j, \quad \text{for all} \quad t > 0, \ i = 1, 2, \ldots, n.
\]

Otherwise, there exist \( i \in \{ 1, 2, \ldots, n \} \) and \( t = \{ 1, 2, \ldots, n \} \) such that

\[
V_i(t) = m_i \xi_j, \quad \text{and} \quad V_j(t) < m_j \xi_j, \quad \text{for all} \quad t \in (\xi_j, t), \ j = 1, 2, \ldots, n.
\]

It follows from (2.8) that

\[
V_i(t) - m_i \xi_j = 0 \quad \text{and} \quad V_j(t) - m_j \xi_j < 0, \quad \forall t \in (\xi_j, t), \ j = 1, 2, \ldots, n.
\]

Together with (2.9) and (2.12), we obtain

\[
0 \leq D^+(V_i(t) - m_i \xi_j) = D^+(V_j(t) - m_j \xi_j)
\]

\[
\leq \sum_{j=1}^{n} b_j(t) \int \mathcal{K}_j(s) g_j(y_j(t-s) + x_j(t-s)) \, ds + \int \mathcal{K}_j(s) g_j(y_j(t-s) + x_j^*(t-s)) \, ds
\]

where \( i = 1, 2, \ldots, n \). Let \( m > 1 \) denote an arbitrary real number such that

\[
m_i \xi_j > \left\| \varphi - \varphi^* \right\| = \sup_{-\infty < s < 0} \max_{1 \leq i \leq n} \varphi_j(s) - \varphi_j^*(s), \quad i = 1, 2, \ldots, n.
\]

It following from (2.8) that

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V_i(t) = \left| y_j(t) \right| e^{\lambda t} < m_i \xi_j, \quad \text{for all} \quad t > 0, \ i = 1, 2, \ldots, n.
\]

We claim that

\[
V_i(t) = \left| y_j(t) \right| e^{\lambda t} < m_i \xi_j, \quad \text{for all} \quad t > 0, \ i = 1, 2, \ldots, n.
\]

Thus,

\[
0 \leq \sum_{j=1}^{n} b_j(t) \int \mathcal{K}_j(s) g_j(y_j(t-s) + x_j(t-s)) \, ds + \int \mathcal{K}_j(s) g_j(y_j(t-s) + x_j^*(t-s)) \, ds
\]

\[
\leq \sum_{j=1}^{n} b_j(t) \int \mathcal{K}_j(s) g_j(y_j(t-s) + x_j^*(t-s)) \, ds + \int \mathcal{K}_j(s) g_j(y_j(t-s) + x_j^*(t-s)) \, ds
\]

where \( i = 1, 2, \ldots, n \). Let \( m > 1 \) denote an arbitrary real number such that

\[
m_i \xi_j > \left\| \varphi - \varphi^* \right\| = \sup_{-\infty < s < 0} \max_{1 \leq i \leq n} \varphi_j(s) - \varphi_j^*(s), \quad i = 1, 2, \ldots, n.
\]

It following from (2.8) that

\[
V_i(t) = \left| y_j(t) \right| e^{\lambda t} < m_i \xi_j, \quad \text{for all} \quad t > 0, \ i = 1, 2, \ldots, n.
\]

We claim that

\[
V_i(t) = \left| y_j(t) \right| e^{\lambda t} < m_i \xi_j, \quad \text{for all} \quad t > 0, \ i = 1, 2, \ldots, n.
\]
3. Results
The following is our main result.

**Theorem 3.1.** Suppose that (T0) - (T4) are satisfied. Then system (1.1) has exactly

a T-anti-periodic solution \( x^*(t) \). Moreover, \( x^*(t) \) is globally exponentially stable.

**Proof.** Let \( v(t) = (v_1(t), v_2(t), \ldots, v_n(t))^T \) be a solution of system (1.1) with initial conditions

\[ v_i(s) = \varphi_i(s), \]

\[ \left| \varphi_i(s) \right| < \zeta_i \frac{1}{\beta}, s \in (-\infty, 0], i = 1, 2, \ldots, n. \]  

(3.1)

According to Remark 2.1, \( v(t) \) exists on \((-\infty, 0]\). Moreover, by Lemma 2.1, the solution \( v(t) \) is bounded and

\[ \left| v_i(t) \right| < \zeta_i \frac{1}{\beta}, \text{ for all } t \in R, i = 1, 2, \ldots, n. \]  

(3.2)

Form (2.1)-(2.3), we have

\[ ((-1)^{k+1}v_i(t+(k+1)T))' = (-1)^{k+1}v_i(t+(k+1)T) \]

\[ = -d_i(t)h_i(t, (-1)^{k+1}v_i(t+(k+1)T)) + \sum_{j \neq i} a_{ij}(t)f_{ij}(v_j(t+(k+1)T) - \tau_i(t)) + \sum_{j \neq i} b_{ij}(t)K_j(s)g_j(v_j(t-s))ds + I_i(t), \]

\[ i = 1, 2, \ldots, n. \]

Thus, for any natural number \( k, (-1)^{k+1}v_i(t+(k+1)T) \)

are the solution of system (1.1) on \( R \). Then, by Lemma 2.2, there exists a constant \( Q > 0 \) such that

\[ \left| (-1)^{k+1}v_i(t+(k+1)T) - (-1)^{k}v_i(t) \right| \leq Qe^{-\lambda(T-t)} \sup_{t \leq s \leq T} \left| v_i(s) \right| \]

\[ \leq 2e^{-\lambda(T-t)}Q \max_{1 \leq i \leq n} \left\{ \zeta_i \frac{1}{\beta} \right\}, \text{ for all } t+KT > 0, i = 1, 2, \ldots, n. \]  

(3.3)

Thus, for any natural number \( \rho \), we obtain

\[ ((-1)^{\rho+1}v_i(t+(\rho+1)T))' = v_i(t) + \sum_{k \geq 1} \left| (-1)^{\rho+1}v_i(t+(\rho+1)T) - (-1)^{\rho}v_i(t+\rho T) \right| \]

where \( i = 1, 2, \ldots, n. \)

In view of (3.4), we can choose a sufficiently large constant \( P > 0 \) and a positive constant \( \alpha > 0 \) such that

\[ \left| (-1)^{k+1}v_i(t+(k+1)T) - (-1)^{k}v_i(t+kT) \right| \leq \alpha(e^{-\lambda T})^k = \alpha e^{-\lambda k}, \]

\[ \forall k > P, i = 1, 2, \ldots, n. \]  

(3.6)

on any compact set of \( R \). It follows from (3.5),(3.6) and

(3.7) that \((-1)^{\rho+1}v_i(t+\rho T) \)

uniformly converges to a continuums function \( x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T \) on any compact set of \( R \).

Now we will show that \( x^*(t) \) is T-anti-periodic solution of system (1.1). First, \( x^*(t) \) is T-periodic, since,

\[ x^*(t) = \lim_{p \to \infty}(-1)^{p+1}v_i(t+(p+1)T) = - \lim_{p \to \infty}(-1)^{p+1}v_i(t+\rho T) = -x^*(t) \]

Next, we prove that \( x^*(t) \) is a solution of system (1.1). In fact, together with the continuity of the right side of system (1.1) and (3.3) implies that

\[ \{((-1)^{p+1}v_i(t+(p+1)T))' \} \]

uniformly converges to a continuous function on any compact set of \( R \). Thus, letting \( p \to \infty \), we obtain

\[ \frac{d}{dt}[x_i(t)'] = -d_i(t)h_i(t, x_i(t')) + \sum_{j \neq i} a_{ij}(t)f_{ij}(x_j(t'-\tau_i(t))) + \sum_{j \neq i} b_{ij}(t)K_j(s)g_j(x_j(t'-s))ds + I_i(t), \]

\[ i = 1, 2, \ldots, n. \]  

(3.7)

Therefore, \( x^*(t) \) is a solution of (1.1). Then, by Lemma 2.2, we can prove that \( x^*(t) \) is globally exponentially stable. This completes the proof.

4. An example
In this section, we will give an example illustrate the feasibilities and effectiveness of our results obtained in section 3.

Let \( n = 2 \). Consider the following cellular neural networks

\[ x_i(t) = (1+T\sin t) + \sum_{s=0}^{1} \frac{\sin s}{8} x_i(t-s), \]

\[ + \sum_{s=0}^{1} \frac{\cos s}{8} x_i(t-s), \]

\[ x_i(t) = (1+T\cos t) + \sum_{s=0}^{1} \frac{\sin s}{8} x_i(t-s), \]

\[ + \sum_{s=0}^{1} \frac{\cos s}{8} x_i(t-s), \]

(4.1)

where

\[ d_1(t) = d_2(t) = 1, h_1(t, u) = 1 + |\sin u|, h_2(t, u) = 1 + |\cos u|, \]

\[ a_{11}(t) = b_{11}(t) = \frac{1}{8} |\sin t|, \]

\[ a_{21}(t) = b_{21}(t) = \frac{1}{8} |\cos t|, a_{22}(t) = b_{22}(t) = \frac{1}{2} |\cos t|, a_{12}(t) = b_{12}(t) = \frac{1}{8} |\cos t|, \]
$$f(u) = f(u) = g(u) = g(u) = u, K_i(s) = K_i(s) = \sin x e^{s}, K_i(s) = \cos x e^{s},$$

$$I_i(t) = -\sin t, I_i(t) = \sin t, \tau_{i1}(t) = 2, \tau_{i2}(t) = 1, \tau_{12}(t) = 6, \tau_{22}(t) = 4.$$ 

Note that

$$h_1 = h_2 = 1, F_1 = F_2 = 1, G_1 = G_2 = 1, a_{11} = a_{12} = a_{22} = b_{21} = b_{22} = \frac{1}{8},$$

$$a_{21} = \frac{1}{2}, a_{22} = \frac{1}{2},$$

$$\bar{I}_1 = \bar{I}_2 = 1, \int_{0}^{\infty} K_i(s) ds \leq k_{ij} = 1, i, j = 1, 2.$$ 

Therefore, it follows from the theory of M-matrix in [16] that there exist constant $\beta = \frac{1}{6} > 0$, and $\zeta_1 = \zeta_2 = 1$ such that for all $t > 0$, there holds

$$-d_i(t) h_i \zeta_i + \sum_{j=1}^{3} a_{i1} F_j \zeta_j + \sum_{j=1}^{3} b_{i1} G_j \zeta_j k_{i1} < -\beta = - \frac{1}{6} < 0,$$

$$-d_i(t) h_i \zeta_i + \sum_{j=1}^{3} a_{i2} F_j \zeta_j + \sum_{j=1}^{3} b_{i2} G_j \zeta_j k_{i2} < -\beta = - \frac{1}{6} < 0,$$

where $i = 1, 2$, which implies that system (4.1) satisfies all the conditions in Theorem 3.1. Hence, system (4.1) has exactly one $T$-anti-periodic solution. Moreover, the $T$-anti-periodic solution is globally exponentially stable.

**Conclusions**

In this paper, cellular neural networks with time-varying delays and continuously distributed delays have been studied. New sufficient conditions for the existence and exponential stability of anti-periodic solutions have been established which extend and improve some previously known results.

**References**