BM-algebras defined by bipolar-valued sets

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Abstract

In this note, by using the concept of Bipolar-valued fuzzy set, the notion of bipolar-valued fuzzy BM-algebra is introduced. Moreover, the notions of (strong) negative s-cut (strong) positive t-cut are introduced and the relationship between these notions and crisp sub-algebras are studied.

Keywords: BM-algebra, Bipolar-valued fuzzy sets, Negative s-cut, (strong) positive t-cut.

Mathematics Subject Classification: 06F35, 03G25, 94D05.

Introduction

Imai and Iseki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras (Iseki & Tanaka, 1978; Iseki, 1980). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Hu and Li (1983 & 1985) introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras.

Neggers and Kim (1999) introduced the notion of d–algebras which is another generalization of BCK-algebras, and also they introduced the notion of B-algebras (Neggers & Kim, 2002a). Moreover, Jun et al. (1998) introduced a new notion, called a BH-algebra, which is a generalization of BCH/BCI/BCK-algebras (Meng & Jun, 1994). Walendziak (2006) obtained another equivalent axiom for B-algebra. Kim et al. (2004) introduced the notion a (pre-) Coxeter algebra and showed that Coxeter algebra is equivalent to an Abelian group all of whose elements have order 2, i.e., a Boolean group. Kim & Kim (2006) introduced the notion of a BM-algebra which is a specialization of B-algebras (Neggers & Kim, 2002b).

Zadeh (1965) introduced the notion of a fuzzy subset of a set; fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains. There have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, soft sets etc.

Lee (2000) introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0,1] to [-1,1]. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0,1] indicates that elements somewhat satisfy the property, and the membership degree [-1,0) indicates that elements somewhat satisfy the implicit counter-property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other (Lee, 2000; Lee, 2004).

Now, in this note we use the notion of Bipolar-valued fuzzy set to establish the notion of bipolar-valued fuzzy BM-algebras; then we obtain some-related which have been mentioned in the abstract.

Preliminary

In this section, we present now some preliminaries on the theory of bipolar-valued fuzzy set. In his pioneer work (Zadeh, 1965), Zadeh proposed the theory of fuzzy sets. Since then it has been applied in wide varieties of fields like Computer Science, Management Science, Medical Sciences, Engineering problems etc. to list a few only.

Definition 1: (Lee 2000) Let G be a nonempty set. A bipolar-valued fuzzy set B in G is an object having the form

$$B = \{(x, \mu^+(x), \nu^-(x)) | x \in G\}$$

Where $\mu^+: G \rightarrow [0,1]$ and $\nu^- : G \rightarrow [-1,0]$ are mappings.

The positive membership degree $\mu^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set

$$B = \{(x, \mu^+(x), \nu^-(x)) | x \in G\}$$

and the negative membership degree $\nu^-(x)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued fuzzy set

$$B = \{(x, \mu^+(x), \nu^-(x)) | x \in G\}$$

If $\mu^+(x) \neq 0$ and $\nu^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for

$$B = \{(x, \mu^+(x), \nu^-(x)) | x \in G\}$$

If $\mu^+(x) = 0$ and $\nu^-(x) \neq 0$, it is the situation that x does not satisfy the property of

$$B = \{(x, \mu^+(x), \nu^-(x)) | x \in G\}$$

But somewhat satisfies the counter-property of

$$B = \{(x, \mu^+(x), \nu^-(x)) | x \in G\}$$
It is possible for an element $x$ to be such that $\mu^+(x) \neq 0$ and $\nu^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of $G$. For the sake of simplicity, we shall use the symbol $B = (\mu^+, \nu^-)$ for the bipolar-valued fuzzy set $B = \left\{ (x, \mu^+(x), \nu^-(x)) \mid x \in G \right\}$.

**Definition 2.** (Kim & Kim, 2006) Let $X$ be a non-empty set with a binary operation "*" and a constant "0". Then $(X, *, 0)$ is called a BM-algebra if it satisfies the following conditions:

1. $x * 0 = x$
2. $(x * y) * z = (x * z) * y$
3. $x * y = 0$ if and only if $y * x = 0$
4. $x * y = 0$ for all $x, y, z \in X$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$.

**Proposition 3.** (Kim & Kim, 2006) Let $X$ be a BM-algebra. Then the followings hold:

1. $x * x = 0$
2. $0 * (0 * x) = x$
3. $0 * (x * y) = y * x$
4. $(x * z) * (y * z) = x * y$
5. $x * y = 0$ if and only if $y * x = 0$
6. $(x * y) * z = (x * z) * y$

A non-empty subset $S$ of $X$ is called a sub-algebra of $X$ if $x * y \in S$ for all $x, y \in S$.

**Definition 4.** (Borumand Saeid, 2010) Let $\mu$ be a fuzzy set in a BM-algebra. Then $\mu$ is called a fuzzy BM-sub-algebra of $X$ if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in X$.

**Bipolar-valued fuzzy subalgebras of BM-algebras**

From now on $(X, *, 0)$ is a BM-algebra, unless otherwise is stated.

**Definition 1:** A bipolar-valued fuzzy set $B = (\mu^+, \nu^-)$ is said to be a bipolar-valued fuzzy sub-algebra of a BM-algebra $X$ if it satisfies the following conditions:

1. $\mu^+(x * y) \geq \min\{\mu^+(x), \mu^+(y)\}$
2. $\nu^-(x * y) \leq \max\{\nu^-(x), \nu^-(y)\}$

for all $x, y \in X$.

**Example 2:** Consider a BM-algebra $X = \{0, 1, 2\}$ with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
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<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $B = (\mu^+, \nu^-)$ be a bipolar-valued fuzzy set in $X$ with the mappings $\mu^+$ and $\nu^-$ defined by:

$$\mu^+(x) = \begin{cases} 0.7 & \text{if } x = 0 \\ 0.3 & \text{if } x \neq 0 \end{cases}$$

and

$$\nu^-(x) = \begin{cases} -0.4 & \text{if } x = 0 \\ -0.2 & \text{if } x \neq 0 \end{cases}$$

It is routine to verify that $B$ is a bipolar-valued fuzzy sub-algebra of $X$.

**Lemma 3:** If $B$ is a bipolar-valued fuzzy sub-algebra of $X$, then $\mu^+(0) \geq \mu^+(x)$ and $\nu^-(0) \leq \nu^-(x)$, for all $x \in X$.

**Proposition 4.** Let $B$ be a bipolar-valued fuzzy sub-algebra of $X$, and let $n \in \mathbb{N}$.

1. $\mu^+ (\prod_{k=0}^{n} x * x) \geq \mu^+ (x)$
2. $\nu^- (\prod_{k=0}^{n} x * x) \leq \nu^- (x)$

where $\prod_{k=0}^{n} x * x = x * x * \cdots * x$.

**Proof.** Let $x \in X$ and assume that $n$ is odd. Then $n = 2k - 1$ for some positive integer $k$. We prove by induction, definition and above lemma imply that $\mu^+ (x * x) = \mu^+ (0) \geq \mu^+ (x)$.

Now suppose that $\mu^+ (\prod_{k=1}^{2(k+1)} x * x) \geq \mu^+ (x)$. Then by assumption

$$\mu^+ (\prod_{k=1}^{2(k+1)} x * x) = \mu^+ (\prod_{k=1}^{2k+1} x * x)$$

$$= \mu^+ (\prod_{k=1}^{2k} x * (x * x))$$

$$\geq \mu^+ (x)$$

also

$$\nu^- (\prod_{k=1}^{2(k+1)} x * x) = \nu^- (\prod_{k=1}^{2k+1} x * x)$$
A fuzzy set $\mu$ of $X$ is called anti fuzzy sub-algebra of $X$, if $\mu(x \ast y) \leq \max\{\mu(x), \mu(y)\}$, for all $x, y \in X$.

Proposition 7: A bipolar-valued fuzzy set $B$ of $X$ is a bipolar-valued fuzzy sub-algebra of $X$ if and only if $\mu^+$ is a fuzzy sub-algebras and $\nu^-$ is an anti fuzzy sub-algebras of $X$.

Proof: The proof is straightforward.

Definition 8: Let $B = (\mu^+, \nu^-)$ be a bipolar-valued fuzzy set and $(s, t) \in [-1,0] \times [0,1]$.

The sets $B^+_s = \{ x \in X \mid \mu^+(x) \geq t \}$ and $B^-_s = \{ x \in G \mid \nu^-(x) \leq s \}$, which are called positive t-cut of $B = (\mu^+, \nu^-)$ and negative s-cut of $B = (\mu^+, \nu^-)$, respectively.

The set $X^{(t, s)}_B = \{ x \in X \mid \mu^+(x) \geq t, \nu^-(x) \leq s \}$ is called an $(s, t)$-level subset of $B$.

The set $X^{(s, t)}_B = \{ x \in X \mid \mu^+(x) > t, \nu^-(x) < s \}$ is called a strong $(s, t)$-level subset of $B$.

The set of all $(s, t) \in \text{Im}(\mu^+) \times \text{Im}(\nu^-)$ is called the image of $B = (\mu^+, \nu^-)$.

Theorem 9: Let $B$ be a bipolar-valued fuzzy subset of $X$ such that the least upper bound to $\text{Im}(\mu^+)$ and the greatest lower bound $s_0$ of $\text{Im}(\nu^-)$ exist. Then the following conditions are equivalent:

(i) $B$ is a bipolar-valued fuzzy sub-algebra of $X$,

(ii) For all $(s, t) \in \text{Im}(\nu^-) \times \text{Im}(\mu^+)$, the nonempty strong level subset $X^{(t, s)}_B$ of $B$ is a (crisp) sub-algebra of $X$.

(iii) For all $(s, t) \in \text{Im}(\nu^-) \times \text{Im}(\mu^+) \setminus (s_0, t_0)$, the nonempty strong level subset $X^{(s, t)}_B$ of $B$ is a (crisp) sub-algebra of $X$. 

\begin{align*}
\mu^+ &= \bigvee_{i=1}^{2^{k-1}} \mu^+ \left( \bigvee_{i=1}^{2^{k-1}} x \ast (x \ast x) \right) \\
\mu^- &= \bigvee_{i=1}^{2^{k-1}} \mu^- \left( \bigvee_{i=1}^{2^{k-1}} x \ast x \right)
\end{align*}

Which proves (i). Similarly we can prove (ii).

Theorem 5: Let $B$ be a bipolar-valued fuzzy sub-algebra of $X$. If there exists a sequence $\{x_n\}$ in $X$, such that

\[
\lim_{n \to \infty} \mu^+(x_n) = 1 \quad \text{and} \quad \lim_{n \to \infty} \mu^-(x_n) = -1.
\]

Then $\mu^+(0) = 1$ and $\mu^-(0) = -1$.

Proof: By above lemma we have $\mu^+(0) \geq \mu^+(x)$, for all $x \in X$, thus $\mu^+(0) \geq \mu^+(x)$, for every positive integer $n$. Consider

$1 \geq \mu^+(0) \geq \lim \mu^+(x_n) = 1$

Hence $\mu^+(0) = 1$ and similarly we have $\mu^-(0) = -1$.

Theorem 6: The family of bipolar-valued fuzzy sub-algebras of $X$ forms a complete distributive lattice under the ordering of bipolar-valued fuzzy set inclusion $\subseteq$.

Proof: Let $\{B_i \mid i \in I\}$ be a family of bipolar-valued fuzzy sub-algebras of $X$. Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering in $[0, 1]$, it is sufficient to show that $\bigcap B_i = \left( \bigvee \mu^+_i \land \bigwedge \nu^-_i \right)$ is a bipolar-valued fuzzy sub-algebra of $X$. Let $x \in X$. Then

\[
(\bigvee \mu^+_i) (x \ast y) = \sup \left\{ \mu^+_i(x \ast y) \mid i \in I \right\}
\]

\[
\geq \sup \left\{ \max \left\{ \mu^+_i(x), \mu^+_i(y) \right\} \mid i \in I \right\}
\]

\[
= \max \left\{ \sup \left\{ \mu^+_i(x) \mid i \in I \right\}, \sup \left\{ \mu^+_i(y) \mid i \in I \right\} \right\}
\]

\[
= \max \left\{ \bigvee \mu^+_i(x), \bigvee \mu^+_i(y) \right\}.
\]

Also we have

\[
(\bigwedge \nu^-_i) (x \ast y) = \inf \left\{ \nu^-_i(x \ast y) \mid i \in I \right\}
\]

\[
\leq \inf \left\{ \min \left\{ \nu^-_i(x), \nu^-_i(y) \right\} \mid i \in I \right\}
\]

\[
= \min \left\{ \inf \left\{ \nu^-_i(x) \mid i \in I \right\}, \inf \left\{ \nu^-_i(y) \mid i \in I \right\} \right\}
\]

\[
= \min \left\{ \bigwedge \nu^-_i(x), \bigwedge \nu^-_i(y) \right\}.
\]

Hence $\bigcap B_i = \left( \bigvee \mu^+_i \land \bigwedge \nu^-_i \right)$ is a bipolar-valued fuzzy sub-algebra of $X$. 

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“BM- algebra”
http://www.indjst.org
(iv) For all \((s,t) \in [-1,0] \times [0,1]\), the nonempty strong level subset \(S X_B^{(s,t)}\) of \(B\) is a (crisp) sub-algebra of \(X\).

(v) For all \((s,t) \in [-1,0] \times [0,1]\), the non-empty level subset \(S X_B^{(s,t)}\) of \(B\) is a (crisp) sub-algebra of \(X\).

Proof: 

\((i \rightarrow iv)\) Let \(B\) be a bipolar-valued fuzzy sub-algebra of \(X\), \((s,t) \in [-1,0] \times [0,1]\) and \(x, y \in S X_B^{(s,t)}\). Then we have

\[\mu^+(x * y) \geq \min \{\mu^+(x), \mu^+(y)\} > \min \{t, t\} = t\]

and

\[v^-(x * y) \leq \max \{v^-(x), v^-(y)\} < \max \{s, s\} = s,\]

Thus \(x * y \in S X_B^{(s,t)}\). Hence \(S X_B^{(s,t)}\) is a (crisp) sub-algebra of \(X\).

\((iv \rightarrow iii)\) It is clear.

\((iii \rightarrow ii)\) Let \((s,t) \in Im(\mu^+) \times Im(v^-)\). Then \(X_B^{(s,t)}\) is nonempty. Since \(X_B^{(s,t)} = \bigcap_{1 < \alpha < \mu} S X_B^{(\mu, \alpha)}\), where \(\beta \in \text{Im}(\mu^+) \setminus t_0\) and \(\alpha \in \text{Im}(v^-) \setminus s_0\). Then by (iii) we get that \(X_B^{(s,t)}\) is a (crisp) sub-algebra of \(X\).

\((ii \rightarrow v)\) Let \((s,t) \in [-1,0] \times [0,1]\) and \(X_B^{(s,t)}\) be nonempty. Suppose that \(x, y \in X_B^{(s,t)}\). Let

\[\alpha = \min \{\mu^+(x), \mu^+(y)\}\]

and

\[\beta = \max \{v^-(x), v^-(y)\}\]

It is clear that \(\alpha \geq s\) and \(\beta \leq t\). Thus \(x, y \in X_B^{(s,t)}\) and \(\alpha \in \text{Im}(\mu^+)\) and \(\beta \in \text{Im}(v^-)\). Hence \(x * y \in X_B^{(s,t)}\). Then we have

\[\mu^+(x * y) \geq \min \{\mu^+(x), \mu^+(y)\} > \min \{\alpha, \alpha\} = \alpha \geq s\]

and

\[v^-(x * y) \leq \max \{v^-(x), v^-(y)\} \leq \max \{\beta, \beta\} = \beta \leq t.\]

Therefore \(x * y \in X_B^{(s,t)}\). Then \(X_B^{(s,t)}\) is a (crisp) sub-algebra of \(X\).

\((v \rightarrow i)\) Assume that the nonempty set \(X_B^{(s,t)}\) is a (crisp) sub-algebra of \(X\), for any \((s,t) \in [-1,0] \times [0,1]\).

In contrary, let \(x_0, y_0 \in X\) be such that

\[\mu^+(x_0 * y_0) < \min \{\mu^+(x_0), \mu^+(y_0)\}\]

and

\[v^-(x_0 * y_0) > \max \{v^-(x_0), v^-(y_0)\}\]

Let

\[\lambda = \frac{1}{2} \left( \mu^+(x_0 * y_0) + \min \{\mu^+(x_0), \mu^+(y_0)\} \right)\]

and

\[\gamma = \frac{1}{2} \left( v^-(x_0 * y_0) + \max \{v^-(x_0), v^-(y_0)\} \right)\]

Hence

\[\alpha > \lambda_1 = \frac{1}{2} (\lambda + \min \{\alpha, \beta\}) > \lambda\]

\[v > \nu_1 = \frac{1}{2} (v + \max \{\theta, \gamma\}) > \theta.\]

Thus

\[\min \{\alpha, \beta\} > \lambda_1 > \lambda = \mu^+(x_0 * y_0)\]

\[\max \{\theta, \gamma\} < \nu_1 < v = v^-(x_0 * y_0),\]

so that \((x_0 * y_0) \not\in X_B^{(\lambda_1, \nu_1)}\) which is a contradiction, since

\[\mu^+(x_0) = \alpha \geq \min \{\alpha, \beta\} > \lambda_1\]

\[\mu^+(y_0) = \beta \geq \min \{\alpha, \beta\} > \lambda_1\]

and

\[v^-(x_0) = \theta \leq \max \{\gamma, \theta\} < \nu_1\]

\[v^-(y_0) = \gamma \leq \max \{\gamma, \theta\} < \nu_1,\]

imply that \((x_0 * y_0) \in X_B^{(\lambda_1, \nu_1)}\). Thus
$\mu^+(x \ast y) \geq \min\{\mu^+(x), \mu^+(y)\}$

And

$\nu^-(x \ast y) \leq \max\{\nu^-(x), \nu^-(y)\}$

For all $x, y \in X$. Now the proof is completed.

Theorem 10: Each sub-algebra of $X$ is a level sub-algebra of a bipolar-valued fuzzy sub-algebra of $X$.

Proof: Let $Y$ be subalgebra of $X$ and $B$ be a bipolar-valued fuzzy subset of $X$ which is defined by:

$$\mu^+(x) = \begin{cases} \alpha & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases}$$

$$\nu^-(x) = \begin{cases} \beta & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases}$$

Where $\alpha \in [0,1]$ and $\beta \in [-1,0]$. It is clear that $X^B_{\mu^+(\cdot)} \subseteq Y$. Let $x, y \in X$. We consider the following cases:

Case 1) If $x, y \in Y$, then $x \ast y \in Y$, therefore

$$\mu^+(x \ast y) = \alpha = \min\{\alpha, \alpha\} = \min\{\mu^+(x), \mu^+(y)\}$$

and

$$\nu^-(x \ast y) = \beta = \max\{\beta, \beta\} = \max\{\nu^-(x), \nu^-(y)\}$$

Case 2) If $x, y \notin Y$, then $\mu^+(x) = 0 = \mu^+(y)$ and $\nu^-(x) = 0 = \nu^-(y)$ and so

$$\mu^+(x \ast y) \geq 0 = \min\{0, 0\} = \min\{\mu^+(x), \mu^+(y)\}$$

and

$$\nu^-(x \ast y) \leq 0 = \max\{0, 0\} = \max\{\nu^-(x), \nu^-(y)\}$$

Case 3) If $x \in Y$ and $y \notin Y$, then

$$\mu^+(y) = 0 = \nu^-(y), \mu^+(x) = \alpha$$

and $\nu^-(x) = \beta$. Thus

$$\mu^+(x \ast y) \geq 0 = \min\{\mu^+(x), \mu^+(y)\}$$

and

$$\nu^-(x \ast y) \leq 0 = \max\{\nu^-(x), \nu^-(y)\}$$

Case 4) If $x \notin Y$ and $y \in Y$, then by the same argument as in case 3, we can conclude the results.

Therefore $B$ is a bipolar-valued fuzzy sub-algebra of $X$.

Theorem 11: Let $S$ be a subset of $X$ and $B$ be a bipolar-valued subset of $X$ which is given in the proof of Theorem 3.10. If $B$ is a bipolar-valued fuzzy sub-algebra of $X$ then $S$ is a sub-algebra of $X$.

Proof: Let $B$ be a bipolar-valued fuzzy sub-algebra of $X$ and $x, y \in S$. Then $\mu^+(x) = \alpha = \mu^+(y)$ and $\nu^-(x) = \beta = \nu^-(y)$, thus

$$\mu^+(x \ast y) \geq \min\{\mu^+(x), \mu^+(y)\} = \min\{\alpha, \alpha\} = \alpha$$

And

$$\nu^-(x \ast y) \leq \max\{\nu^-(x), \nu^-(y)\} = \max\{\beta, \beta\} = \beta$$

Which implies that $x \ast y \in S$.

Now we generalize the Theorem 3.10.

Theorem 12: Let $X$ be a $BM$-algebra. Then for any chain of sub-algebras $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_r = X$

There exists a bipolar-valued fuzzy sub-algebra $B$ of $X$ whose level sub-algebras are exactly the sub-algebras of this chain.

Proof: Consider the following sets of numbers $p_0 > p_1 > \cdots > p_r$

and

$q_0 < q_1 < \cdots < q_r$

where each $p_i \in [0,1]$ and $q_i \in [-1,0]$. Define $\mu^+$ and $\nu^-$ by:

$$\mu^+(A_i \setminus A_{i-1}) = p_i$$

for all $0 < i \leq r$, and $\mu^+(A_0) = p_0$, and

$$\nu^-(A_i \setminus A_{i-1}) = q_i$$

for all $0 < i \leq r$ and $\nu^-(A_0) = q_0$.

We prove that $B = (\mu^+, \nu^-)$ is a bipolar-valued fuzzy sub-algebra of $X$. Let $x, y \in X$, we consider the following cases:

Case 1) If $x, y \in A_i \setminus A_{i-1}$, then $\mu^+(x) = p_i = \mu^+(y)$ and $\nu^-(x) = q_i = \nu^-(y)$. Since $A_i$ is a sub-algebra thus $x \ast y \in A_i$, so $x \ast y \in A_i \setminus A_{i-1}$ or $x \ast y \in A_{i-1}$ and in each of then we have

$$\mu^+(x \ast y) \geq p_i = \min\{\mu^+(x), \mu^+(y)\}$$

and

$$\nu^-(x \ast y) \leq q_i = \max\{\nu^-(x), \nu^-(y)\}$$

Case 2) If $x \in A_i \setminus A_{i-1}$ and $y \in A_j \setminus A_{j-1}$, where $i < j$. Then $\mu^+(x) = p_i, \mu^+(y) = p_j, \nu^-(x) = q_i$, and $\nu^-(y) = q_j$. Since $A_j \subseteq A_i$ and $A_i$ is a sub-algebra of $X$, then $x \ast y \in A_i$. Hence
\[ \mu^+(x \ast y) \geq p_i = \min\{\mu^+(x), \mu^+(y)\} \]
and
\[ v^-(x \ast y) \leq q_i = \max\{v^-(x), v^-(y)\} \]
It is clear that \( \text{Im}(\mu^+) = \{p_0, p_1, \ldots, p_r\} \) and \( \text{Im}(v^-) = \{q_0, q_1, \ldots, q_r\} \), therefore the level sub-algebras of \( \mu^+ \) and \( v^- \) are given by the chain of sub-algebras
\[(\mu^+_{p_0}, v^-_{q_0}) \subseteq (\mu^+_{p_1}, v^-_{q_1}) \subseteq \cdots \subseteq (\mu^+_{p_r}, v^-_{q_r}) = X \]
We have
\[(\mu^+_{p_i}, v^-_{q_i}) = \{x \in X \mid \mu^+(x) \geq p_i, v^-(x) \leq q_i\} = A_i \]
It is clear that \( A_i \subseteq (\mu^+_{p_i}, v^-_{q_i}) \). Let \( x \in (\mu^+_{p_i}, v^-_{q_i}) \). Then \( \mu^+(x) \geq p_i \) and \( v^-(x) \leq q_i \), thus \( x \in A_j \) for \( j > i \).
So
\[ \mu^+(x) \in \{p_0, p_1, \ldots, p_r\} \]
and
\[ v^-(x) \in \{q_0, q_1, \ldots, q_r\} \]
for \( 0 \leq i \leq r \).

Theorem 13: If \( B = (\mu^+, v^-) \) is a bipolar-valued fuzzy sub-algebra of \( X \), then the set
\[ X_B = \{x \in X \mid \mu^+(x) = \mu^+(0), v^-(x) = v^-(0) = v^-(x)\} \]
is a sub-algebra of \( X \).
Proof: Let \( x, y \in X_B \). Then \( \mu^+(x) = \mu^+(0) = \mu^+(y) \) and \( v^-(x) = v^-(0) = v^-(y) \), and so
\[ \mu^+(x \ast y) \geq \min\{\mu^+(x), \mu^+(y)\} \]
and
\[ v^-(x \ast y) \leq \max\{v^-(x), v^-(y)\} \]

By Lemma 3.3, we get that \( \mu^+(x \ast y) = \mu^+(0) \) and \( v^-(x \ast y) = v^-(0) \) which means that \( x \ast y \in X_B \).

Theorem 14: Let \( M \) be a subset of \( X \). Suppose the \( N \) is a bipolar-valued fuzzy set of \( X \) defined by:
\[ \mu^+_N(x) = \begin{cases} \alpha & \text{if } \alpha \in M \\ \beta & \text{otherwise} \end{cases} \]
and
\[ v^-_N(x) = \begin{cases} \gamma & \text{if } x \in M \\ \delta & \text{otherwise} \end{cases} \]
For all \( \alpha, \beta \in [0,1] \) and \( \gamma, \delta \in [-1,0] \) with \( \alpha \geq \beta \) and \( \gamma \leq \delta \). Then \( N \) is a bipolar-valued fuzzy sub-algebra if and only if \( M \) is a sub-algebra of \( X \). Moreover, in this case \( X_N = M \).
Proof. Let \( N \) be a bipolar-valued fuzzy sub-algebra. Let \( x, y \in X \) be such that \( x, y \in M \). Then \( \mu^+_N(x \ast y) \geq \min\{\mu^+_N(x), \mu^+_N(y)\} \)
\[ = \min\{\alpha, \alpha\} = \alpha \]
and
\[ v^-_N(x \ast y) \leq \max\{v^-_N(x), v^-_N(y)\} \]
\[ = \min\{\gamma, \gamma\} = \gamma \]
Therefore \( x \ast y \in M \).

Conversely, suppose that \( M \) is sub-algebra of \( X \), let \( x, y \in X \).
If \( x, y \in M \), then \( x \ast y \in M \), thus
\[ \mu^+_N(x \ast y) = \alpha = \min\{\mu^+_N(x), \mu^+_N(y)\} \]
and
\[ v^-_N(x \ast y) = \gamma = \max\{v^-_N(x), v^-_N(y)\} \]
(ii) If \( x \not\in M \) or \( y \not\in M \), then
\[ \mu^+_N(x \ast y) \geq \beta = \min\{\mu^+_N(x), \mu^+_N(y)\} \]
and
\[ v^-_N(x \ast y) \leq \delta = \max\{v^-_N(x), v^-_N(y)\} \]
This shows that \( N \) is a bipolar-valued fuzzy sub-algebra.

Moreover, we have
\[ X_N = \{x \in X \mid \mu^+_N(x) = \mu^+_N(0), v^-_N(x) = v^-_N(0) = v^-_N(0)\} \]
\[ = \{x \in X \mid \mu^+_N(x) = \alpha, v^-_N(x) = \gamma\} = M \]
Definition 15: \( X \) is said to be Artinian if it satisfies the descending chain condition on sub-algebras (simply written as DCC), that is, for every chain
\[ I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots \] of sub-algebras of \( X \), there is a natural number \( i \) such that \( I_i = I_{i+1} = \cdots \).

Theorem 16: Each bipolar fuzzy sub-algebra \( X \) has finite values if and only if \( X \) is Artinian.
Proof. Suppose that each bipolar fuzzy sub-algebra of \( X \) has finite values. If \( X \) is not Artinian then there is a strictly descending chain
of sub-algebras of $X$, where $I_i \supset I_j$ expresses $I_i \supset I_j$ but $I_i \neq I_j$. We now construct the bipolar fuzzy set $B = (\mu^+, \nu^-)$ of $X$ by

$$
\mu^+(x) := \begin{cases} 
\frac{n}{n+1} & \text{if } x \in I_n \setminus I_{n+1}, n = 1, 2, \ldots, \\
1 & \text{if } x \in \bigcap_{n=1}^{\infty} I_n,
\end{cases}
$$

$$
\nu^-(x) := -\mu^+(x).$

We first prove that $B$ is a bipolar fuzzy sub-algebra of $X$. For this purpose, we need to verify that $\mu^+$ is a fuzzy sub-algebra of $X$. We assume that $x, y \in X$. Now, we consider the following cases:

Case 1: $x, y \in I_n \setminus I_{n+1}$. In this case, $x, y \in I_n$ and $x \ast y \in I_n$. Thus

$$
\mu^+(x \ast y) \geq \frac{n}{n+1} = \min \{\mu^+(x), \mu^+(y)\}.
$$

Case 2: $x \in I_n \setminus I_{n+1}$ and $y \in I_m \setminus I_{m+1}$ ($n < m$). In this case, $x, y \in I_n$ and $x \ast y \in I_n$. Thus

$$
\mu^+(x \ast y) \geq \frac{n}{n+1} = \min \{\mu^+(x), \mu^+(y)\}.
$$

Case 3: $x \in I_n \setminus I_{n+1}$ and $y \in I_m \setminus I_{m+1}$ ($n > m$). In this case, $x, y \in I_m$ and $x \ast y \in I_m$. Thus

$$
\mu^+(x \ast y) \geq \frac{m}{m+1} = \min \{\mu^+(x), \mu^+(y)\}.
$$

Therefore $\mu^+$ satisfies (BF1), and so $\mu^+$ is a fuzzy sub-algebra of $X$. This shows that $B$ is a bipolar fuzzy sub-algebra of $X$, but the values of $B$ are infinite, which is a contradiction. Thus $X$ is Artinian.

Conversely, suppose that $X$ is Artinian. If there is a bipolar fuzzy sub-algebra $B = (\mu^+, \nu^-)$ of $X$ with $\text{Im}(B) = +\infty$, then $\text{Im}(\mu^+) = +\infty$ or $\text{Im}(\nu^-) = +\infty$.

Without loss of generality, we may assume that $\text{Im}(\mu^+) = +\infty$. Select $s_i \in \text{Im}(\mu^+)$ ($i = 1, 2, \ldots$) and $s_1 < s_2 < \cdots$. Then $U(\mu^+; s_i)$ ($i = 1, 2, \ldots$) are sub-algebras of $X$ and $U(\mu; s_i) \supset U(\mu^+; s_i) \not\supset U(\mu^+; s_{i+1})$ ($i = 1, 2, \ldots$), which is a contradiction. Similar for $\text{Im}(\nu^-)$. The proof is completed.

Definition 17: A BM-algebra $X$ is said to be Noetherian if every sub-algebra of $X$ is finitely generated. $X$ is said to satisfy the ascending chain condition (briefly, ACC) if for every ascending sequence $I_1 \subseteq I_2 \subseteq \cdots$ of sub-algebras of $X$ there is a natural number $n$ such that $I_i = I_n$ for all $i \geq n$.

Theorem 18. $X$ is Noetherian if and only if for any bipolar-valued fuzzy sub-algebra $B$, the set $\text{Im}(B)$ is a well ordered subset, that is, $\left(\text{Im}(\mu^+), \leq\right)$ and $\left(\text{Im}(\nu^-), \geq\right)$ are well ordered subsets of $[0,1]$ and $[-1,0]$, respectively.

Proof. Suppose that $X$ is Noetherian. For any chain $t_1 > t_2 > \cdots$ of $\text{Im}(\mu^+)$, let $t_0 = \inf \{t_i \mid i = 1, 2, \ldots\}$. Then $I := \{x \in G \mid \mu^+(x) > t_0\}$ is sub-algebra of $X$, and so $I$ is finitely generated. Let $I = (a_1, a_2, \ldots, a_k)$. Then $\mu^+(a_1) \wedge \cdots \wedge \mu^+(a_k)$ is the least element of the chain $t_1 > t_2 > \cdots$. Thus $\left(\text{Im}(\mu^+), \leq\right)$ is a well ordered subset of $[0,1]$. By using the same argument as above, we can easily show that $\left(\text{Im}(\nu^-), \geq\right)$ is a well ordered subset of $[-1,0]$. Therefore, $\text{Im}(B)$ is a well ordered subset.

Conversely, let $\text{Im}(B)$ be well ordered subset. If $X$ is not Noetherian, then there is a strictly ascending sequence of sub-algebras of $X$ such that $I_1 \subseteq I_2 \subseteq \cdots$.

We construct the bipolar fuzzy set $B = (\mu^+, \nu^-)$ of $X$ by

$$
\mu^+(x) := \begin{cases} 
1 & \text{if } x \in I_n \setminus I_{n+1}, n = 1, 2, \ldots, \\
0 & \text{if } a \notin \bigcup_{n=1}^{\infty} I_n,
\end{cases}
$$

$$
\nu^-(x) := -\mu^+(x).$

Where $I_0 = 0$. By using similar method as the necessity part of Theorem 3.16, we can prove that $B$ is a bipolar-valued fuzzy sub-algebra of $X$. Because $\text{Im}(B)$ is not well ordered, which is a contradiction. This completes the proof.

Conclusion
Bipolar-valued fuzzy set is a generalization of fuzzy sets. In the present paper, we have introduced the concept of bipolar-valued fuzzy sub-algebras of BM-algebras and investigated some of their useful properties. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as groups, semigroups, rings, nearrings, semirings (hemirings), lattices and Lie algebras. It is our hope that this work would other foundations for further study of the theory of -algebras.

In our future study of fuzzy structure of BM-algebras may be the following topics should be considered:
To establish a bipolar-valued fuzzy ideals of BM-algebras;
To consider the structure of quotient BM-algebras by using these bipolar-valued fuzzy ideals;
To get more results in bipolar-valued fuzzy BM-algebras and application.

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