An integration procedure for meshless methods using Kriging interpolations

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Abstract
A new method is presented in this paper to evaluate the integrals of the Galerkin weak form in meshless methods without any background mesh. Simple Kriging interpolation method has been used to obtain the weights of the integration points. Then, the integration of the Galerkin weak form is evaluated using the introduced weights of the integration points. In the presented method the required integration points can be the same as nodal points. The numerical results show the efficiency of this integration technique in 2D elasticity problems.

Keywords: Integration of Galerkin weak form, Meshless methods, Weights of integration points, Kriging interpolation method.

1. Introduction
In Galerkin formulations, the evaluation of integrals which appear in the weak form equation is of great importance in computational methods such as the finite element method (FEM) (Zienkiewics & Taylor, 2000) and meshless methods (Liu, 2002; Chen & Liu, 2004). In the FEM, the problem domain is divided into elements and the Gaussian quadrature is used in each element for integration purposes. The FEM has some restrictions, such as the necessity for re-meshing due to element distortion. Meshless methods have been introduced and developed over the past two decades to overcome the FEM difficulties. However, their numerical integrations are not that straightforward as compared with the FEM and many solutions have been proposed to solve this issue (Chen & Liu, 2004; Dolbow & Belytschko, 1999).

The so called, “not truly meshless” methods (Atluri & Shen, 2002) such as the element free Galerkin (EFG) method (Belytschko et al., 1994) and meshless radial point interpolation method (RPIM) (Liu & Wang, 2002) need background mesh to perform the appeared integrals in weak form equation. Using background mesh is a well-known technique and yields accurate results in meshless methods based on the global weak form (Liu & Wang, 2002; Belytschko et al., 1994; Belytschko & Tabbara, 1996; Li & Belytschko, 2001; Ponthot & Belytschko, 1998; Singh & Tanaka, 2006; Krysl & Belytschko, 1995).

Much effort has been invested to obtain “truly meshless” methods, in which mesh is avoided.

One solution for avoiding the background mesh is to use a local weak formulation, which has been used in the meshless local Petrov-Galerkin (MLPG) method (Atluri & Shen, 2002; Gu & Liu, 2001; Gu & Liu, 2001), the local radial point interpolation method (LRPIM) (Gu & Liu, 2001; Liu & Gu, 2001; Liu & Gu, 2002) and Local Kriging (LoKriging) method (Lam et al., 2004; Gu et al., 2007). The meshless methods which use local weak form may be computationally expensive (Liu & Gu, 2003).

Another solution to avoid the background mesh is the nodal integration technique in which the integration over the domain is evaluated at nodes. Beissel and Belytschko (1996) used this technique in the EFG method. However, the accuracy of the nodal integration method is lower than with the original EFG method (Beissel and Belytschko, 1996).

The Monte Carlo integration technique (James, 1980) has also been used as a solution which performs the integration based on both global and local weak forms (Rosca & Leitao, 2008). This technique has been implemented for the EFG and the MLPG methods. However, this integration method, require a large number of integration points (Rosca & Leitao, 2008).

The question is that: “Is there a technique that evaluates the integrals without a background mesh and uses the global weak form?” Also, “Can these integrals be evaluated with a lower number of the integration points than other meshless methods?” The presented method covers both questions simultaneously.

An improvement that is presented in this paper is the usage of the Kriging interpolation method (Cressie, 1993) for obtaining the weights of the integration points. In the presented method, the weak form integration in meshless methods is performed over the entire domain without any background mesh. The arrangement and number of integration points are independent of the nodal points. Numerical examples show that the number of integration points in the presented method can even be the same as the number of nodal points.

Numerical results show the efficiency of this integration method for meshless methods using global weak form in 2D elasticity problems. This method can be extended to other problems such as metal forming processes. However this is beyond the scope of the present paper although this has been done for upsetting and ring compression. These results will be presented...
in near future.

This paper is arranged as follows: In section 2, the theory of Kriging interpolation method is reviewed. In section 3 a new technique for obtaining the weights of integration points is discussed and is numerically compared with standard Gauss quadrature weights. Also as an example integration over a circular domain is performed. In section 4, the governing equations and their discretization for elasticity problems have been presented. Numerical results are presented in section 5. Finally, conclusions are presented in section 6.

2. Kriging interpolation method

Kriging is an interpolation method which in recent decades was widely used in mining engineering. The Kriging interpolation method was first introduced by Matheron (1962) and was developed in subsequent years in the context of geo-statistics problems. Different kinds of Kriging interpolation methods have been used such as the simple Kriging and universal Kriging methods (Cressie, 1993). In the simple Kriging method, the field variable \( u(X_0) \) for an arbitrary point within the problem domain \( X_0 \) is approximated as follows (Gu et al., 2007):

\[
u(x_0) \approx u^h(x_0) = \sum_i \lambda_i u_i = \Lambda^T \hat{U}
\]

where \( n \) is the number of nodes within the influence domain of the point at \( X_0 \), \( u^h(x_0) \) represents the approximation of the field variable \( u(x) \), \( u_i \) is the nodal field variable at the \( i \)th node, \( \lambda_i \) is the Kriging shape function of the \( i \)th node, \( \lambda = [\lambda_1, \lambda_2, ..., \lambda_n]^T \) is the vector of the shape functions and \( \hat{U} = [u_1, u_2, ..., u_n]^T \) is the vector that collects the field variables at nodes.

In 2D space according to the simple Kriging method, these shape functions can be derived as follows:

\[
\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} C_{01} \\ C_{02} \\ \vdots \\ C_{0n} \end{bmatrix}
\]

where, \( C_{ij} = \text{Cov}(h_i, h_j) = ce^{-\frac{(||x_i - x_j||)^2}{a^2}} \), \( \text{Cov}(h_i) \) is the covariance function, and \( h_i = ||x_i - x_j|| \) is the distance between points \( i \) and \( j \). The coefficients \( a \) and \( c \), used from literatures are as follows (Cressie, 1993):

\[
\begin{align*}
a &= \alpha r_{int} \\
c &= 1
\end{align*}
\]

in which \( r_{int} \) is the radius of influence domain.

The simple Kriging method is not appropriate for constructing the shape functions in numerical methods because this method cannot ensure a certain degree of consistency. Therefore, the universal Kriging method is used for construction of the shape functions.

In 2D space according to the universal Kriging method, the field variable \( u(X_0) \) is approximated by Eq. 1, in which \( \lambda_i \) is the shape function of the \( i \)th node and can be derived as follows:

\[
\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} C_{01} \\ C_{02} \\ \vdots \\ C_{0n} \end{bmatrix}
\]

where coefficients \( \eta_i (i=1, ..., n) \) are the Lagrange multipliers.

In this paper, for the first time, the simple Kriging method has been used to obtain the weights of the integration points as will be discussed in section 3. To construct the shape functions and calculate the stiffness matrix, the universal Kriging method has been used.

3. Estimation of the weights of integration points

In this paper, for obtaining the weights of the integration points, simple Kriging method is used. In simple Kriging interpolation method, the integration of the field can be done as follows:

\[
\int_{\Omega} u(x) d\Omega \approx \int_{\Omega} u^h(x) d\Omega = \int_{\Omega} \hat{U} d\Omega
\]

in which, \( \Omega \) is the problem domain and \( V = [\text{Cov}(x_i, x) \cdots \text{Cov}(x_i, x)] \)

where, \( n \) is the number of integration points within the influence domain which is assumed to be the whole domain \( \Omega \), \( x_i \) is the position of the \( i \)th integration point and \( x \) is the space coordinates. \( A \) is defined in Eq. 2, in which \( a \) and \( c \) are assumed to be constants over the whole domain. On the other hand,

\[
\int_{\Omega} u(x) d\Omega \approx \sum w_i u_i = W^T \hat{U}
\]

\( w_i \) is the weight of the \( i \)th integration point. Therefore:

\[
W = A^{-1} \int_{\Omega} V d\Omega
\]

Or

\[
\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} \text{Cov}(x_1, x_1) & \text{Cov}(x_1, x_2) & \cdots & \text{Cov}(x_1, x_n) \\ \text{Cov}(x_2, x_1) & \text{Cov}(x_2, x_2) & \cdots & \text{Cov}(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(x_n, x_1) & \text{Cov}(x_n, x_2) & \cdots & \text{Cov}(x_n, x_n) \end{bmatrix} \begin{bmatrix} \int_{\Omega} \text{Cov}(x_1, x) d\Omega \\ \int_{\Omega} \text{Cov}(x_2, x) d\Omega \\ \vdots \\ \int_{\Omega} \text{Cov}(x_n, x) d\Omega \end{bmatrix}
\]
In 2D problems, the integral terms on the right hand side of Eq. 8 \( \int \text{Cov}(x_i, x) d\Omega \) must be calculated on the problem domain \( \Omega \). Meanwhile, \( \text{Cov}(x_i, x) \) which is evaluated by \( e^{-\frac{d}{a}^2} \) is negligible for domains far from integration point \( x_i \). Therefore, the value of \( \int \text{Cov}(x_i, x) d\Omega \) can be evaluated in a smaller domain. Consequently without losing generality, it can be assumed that \( \int \text{Cov}(x_i, x) d\Omega = \int \text{Cov}(x_i, x) d\Omega \) in which \( \Omega \)is a smaller influence domain than problem domain. The radius of influence domain is assumed to be 2.5 \( a \) in which constant \( a \) is the average of the minimum distance between integration points. A greater radius of influence domain has negligible effect on the results. Evaluation of this integral for integration points far from the boundaries is done precisely. If the influence domain crosses the integration boundary, this integral cannot be obtained directly. Figure 1 shows two integration points near the boundary and their own influence domains. If influence domain \( \Omega \) of any integration point \( x_i \) crosses the boundary, to calculate the \( \int \text{Cov}(x_i, x) d\Omega \) over the influence domain, the radius of the influence domain is subdivided into several pieces. A ring is made on each piece from \( r_i \) to \( r_{i+1} \) as illustrated in Figure 1. The integration over the influence domain can be obtained simply by summing up the integrals over these rings as follows:

\[
\int_{\Omega} \text{Cov}(x_i, x) d\Omega = \int_{\Omega} c_i e^{-\frac{(x-x_i)^2}{a^2}} r dr \theta = \sum_i 2\pi \int_a^r c_i e^{-\frac{(x-x_i)^2}{a^2}} r dr
\]

in which \( i \) is the counter of the rings. If the influence domain crosses the integration boundary, the ratio of \( \frac{\text{dx}_{i+1} - \text{dx}_i}{a} \) is multiplied by \( 2\pi \int_a^r c_i e^{-\frac{(x-x_i)^2}{a^2}} r dr \) for each ring in Eq. 9. The value of \( a \) can be obtained from the geometry of the cut section of the boundary, as shown in Figure 1, on the mean radius \( r_m \).

The integration over the domain is evaluated at the integration points from which their weights are obtained using Eq. 7. The arrangement and number of integration points is independent of the nodal points. Optimization of locations of the integration points is beyond the scope of the present work and be performed in future. However, to obtain more accurate results, an adequate number and arrangement for the integration points can be determined by several trial runs. Different types of arrangements for integration points are examined for 2D elasticity problems and the accuracy of the results is compared in numerical examples.

As an example, the results of the presented method have been compared with the Gauss quadrature results. For a square domain, the integration points are located at the positions of Gauss quadrature points. In Table 1, for a square element with 16 integration points, the weights of the integration points obtained from the present method have been compared with those obtained from the Gauss quadrature method. It is shown, that for a standard element, the results are in a very good agreement with the Gauss quadrature method.

**Table 1. Comparison of the weights of integration points in a square element with 16 integration points using the presented method and Gauss quadrature method**

<table>
<thead>
<tr>
<th>Situation of integration points</th>
<th>Presented method weights</th>
<th>Gauss quadrature weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = \pm 0.339981 ) ( y = \pm 0.339981 ) &amp; ( x = \pm 0.861136 ) ( y = \pm 0.861136 )</td>
<td>0.4253072 0.2268552 0.1209783</td>
<td>0.4252929 0.2268518 0.1210030</td>
</tr>
</tbody>
</table>

The present method can be used in every domain. As another example, let us consider a circular domain with arbitrary integration points which is shown in Figure 2. The positions of these points are as follows:

**Fig.2. A circular domain with integration points**
\[ \Omega : x^2 + y^2 \leq 1 \]
\[ P_{1,2} = 0 , \pm 0.8 \]
\[ P_{3,4} = \pm 0.8 , 0 \]
\[ P_{5,6,7,8} = \pm 0.3 , \pm 0.3 \]

Using the present method, the weights are as follows:
\[ w_{1,2,3,4} = 0.5448 \]
\[ w_{5,6,7,8} = 0.2406 \]

For the above circular domain and the selected integration points and arbitrary test functions shown in Table 2, the integrations are evaluated by Eq. 12 and the relative errors compared with the exact solutions are shown in Table 2.

\[ \int_{x^2+y^2<1} f(x,y) \, dx\,dy = \sum_{i=1}^{8} w_i f(x_i,y_i) \quad (12) \]

As it can be seen from Table 2 error values are very promising. The relative error is calculated as follows:

\[ \text{Error} = \frac{u_{\text{reference}} - u_{\text{numerical}}}{u_{\text{reference}}} \times 100 \quad (13) \]

### Table 2. Comparison of the results using the presented method and exact integration values

<table>
<thead>
<tr>
<th>Arbitrary test function</th>
<th>Integration using the present method</th>
<th>Exact integration</th>
<th>Error%</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x,y) = 3x^2 + y^2 )</td>
<td>3.135840</td>
<td>3.141592</td>
<td>0.183%</td>
</tr>
<tr>
<td>( f(x,y) = 10(x-2)^4 + 5(x-2)^2 + 3(x-2)^2(y-2)^2 + 4(y-2)^4 + 10 )</td>
<td>1016.552</td>
<td>1016.698</td>
<td>0.014%</td>
</tr>
<tr>
<td>( f(x,y) = x^4 - 3x^3y + x^2y^2 - 2xy^3 + y^4 )</td>
<td>0.915987</td>
<td>0.916298</td>
<td>0.034%</td>
</tr>
</tbody>
</table>

In the section 5, the present method is used in 2D elasticity problems.

### 4. Global weak formulation

Consider a linear elastic body in a 2D domain \( \Omega \), bounded by \( \Gamma \) as shown in Figure 3. The solid is assumed to undergo infinitesimal deformations.

**Fig.3.** A linear elastic body in a 2D domain \( \Omega \), bounded by \( \Gamma \)

The equilibrium equation can be written as:

\[ \sigma_{ij} + b_i = 0 \quad \text{in} \ \Omega \quad (14) \]

where \( \sigma_{ij} \) is the stress tensor, which corresponds to the displacement field \( u_i \) and \( b_i \) is the body force. The corresponding boundary conditions are given as follows:

\[ u_i = \tilde{n}_i \quad \text{on} \ \Gamma_s \]
\[ \sigma_{ij} n_i = \tilde{t}_j \quad \text{on} \ \Gamma_n \]

where \( \tilde{n}_i \) and \( \tilde{t}_j \) are the prescribed displacements and tractions, respectively, on the displacement boundary \( \Gamma_s \) and on the traction boundary \( \Gamma_n \), and \( n_i \) is the unit outward normal to the boundary \( \Gamma_n \).

By using the principle of virtual work, the global weak form is obtained as follow (Liu, 2002):

\[ \int_{\Omega} \delta \phi \, \sigma \, d\Omega - \int_{\Omega} \delta \phi \, \mathbf{b} \, d\Omega - \int_{\Gamma_s} \delta \phi \, \mathbf{t} \, d\Gamma = 0 \quad (16) \]

The discretized form of Eq. 16 can be written as:

\[ \mathbf{Ku} = \mathbf{F} \quad (17) \]

where \( \mathbf{u} \) is the displacement vector of the nodal points, \( \mathbf{K} \) is the stiffness matrix, and \( \mathbf{F} \) is the force vector defined as follows:

\[ \mathbf{K}_{ij} = \int_{\Omega} B_i^T \mathbf{D} B_j \, d\Omega \quad (18) \]
\[ \mathbf{F}_i = \int_{\Gamma_s} N_i^T \mathbf{t} \, d\Gamma + \int_{\Gamma_n} N_i^T \mathbf{b} \, d\Omega \quad (19) \]

For plane stress condition the elastic matrix \( \mathbf{D} \) is given as:

\[ D = \frac{E}{1-\nu^2} \left[ \begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{array} \right] \quad (20) \]

in which, \( E \) and \( V \) are Young’s modulus and Poisson’s ratio respectively. \( \mathbf{B}_i \) and \( \mathbf{N}_i \) have been calculated as follows:

\[ \mathbf{B}_i = \left[ \begin{array}{c} \frac{\partial \phi_i}{\partial x} \\ \frac{\partial \phi_i}{\partial y} \\ \frac{\partial \phi_i}{\partial n} \end{array} \right] \quad (21) \]
\[ \mathbf{N}_i = \left[ \begin{array}{c} \phi_i \\ 0 \end{array} \right] \quad (22) \]

The shape functions \( \phi_i \)’s can be calculated from any interpolation function. In this paper the universal Kriging interpolation method is used. Using the universal Kriging interpolation function which possesses the Kronecker delta property, enforcement of the essential boundary conditions is done readily. In this interpolation \( \phi_i \) is the same as \( \lambda_i \) in Eq. 4. For evaluating the inte-
grals in Eq. 18 and Eq. 19, in FEM and some meshless methods, a mesh is required. The present integration method is used for evaluating the integrals and can be implemented for any arbitrary arrangements of integration points and also does not need any background mesh.

5. Numerical examples

To investigate the capabilities of the present method, three numerical examples are solved. In the first example, the bending of a cantilever beam, and several types of arrangements for integration points are investigated, and the accuracies of the results are compared. In the second example, a plate under linear traction is evaluated. In the third example the convergence of the present method is verified in a plate with a hole. For construction of the shape functions, universal Kriging interpolation method with linear base functions is used.

5.1 Bending of a cantilever beam

In the first example, the bending of a cantilever beam is evaluated, loaded at the end as shown in Figure 4. Loading and beam data as follows: \( L=30 \text{m}, D=4 \text{m}, E=200 \text{Gpa}, \nu=0.3 \) and \( P=10^7 \text{N} \). The equation of the exact vertical deflection \( u_y \) is (Timoshenko & Goodier, 1970):

\[
  u_y = -\frac{P}{6EI} \left( 3uy^2x + (4 + 5\nu) \frac{D^2(L - x)}{4} + (2L + x)(L - x)^2 \right)
\]

where the moment of inertia for a beam with rectangular cross section and unit thickness is \( I = \frac{D^4}{12} \) and the shear stress on the cross section of the beam is

\[
  \tau = -\frac{P}{2I} \times \left( \frac{D^2}{4} - y^2 \right)
\]

In this example, various arrangements of integration points are compared. Integration points are divided into two sets, i.e. inner and outer sets. The inner set is chosen to coincide at the inner nodal points and the outer set is selected on the boundaries and in the middle of the nodal points as shown in Figure 5. A regular distribution of integration points is used.

Table 4. The stress concentration factor versus number of nodes

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Present method</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>176</td>
<td>2.73</td>
<td>3.00</td>
</tr>
<tr>
<td>247</td>
<td>2.87</td>
<td></td>
</tr>
<tr>
<td>408</td>
<td>2.96</td>
<td></td>
</tr>
</tbody>
</table>

Fig.5. Positions of inner and outer integration sets and also nodal points. Outer integration set is selected adjacent to the boundaries.

The displacements of points are compared with the exact solution in Figure 6. Maximum relative error in deflection of the points is 5%.

Fig.6. The bending of a cantilever beam. Outer integration set is selected adjacent to the boundaries. Deformations in this figure are scaled up 100 times.

In the second arrangement, the outer set is located at 0.05 units inside the boundaries. Figure 7 shows the positions of the integration points. The corresponding results are shown in Figure 8. Maximum relative error in deflection is 2% in this case.

Then, the distance from the boundary was increased to 0.25 in the next trial. Figure 9 and Figure 10 show arrangement of the new integration points and their corresponding results respectively. Improved results were obtained when the distance between the integration points and boundaries was 0.25, in which the maximum relative error in deflection is 0.54%.

Fig.7. Positions of inner and outer integration sets and also nodal points. Distances between outer set of integration points and the boundaries are 0.05.
5.2 Plate under linear traction

Figure 13 shows a rectangular plate with linear traction. The dimensions of the rectangular plate are \( L=20 \text{m}, D=10 \text{m} \), with unit thickness and mechanical properties are the same as in the previous example.

**Fig. 13. A rectangular plate under a linear traction**

\[
\sigma_z = 1 \text{ MPa}
\]

In the first arrangement of the integration points, the integration points and nodal points are the same. In Figure 14, the nodal points are plotted after the final load. The results have some fluctuations similar to the hour glassing phenomenon which occurs in the FEM when the number of integration points is insufficient (Belytschko et al., 2000).

**Fig. 14. Deformation of a rectangular plate under linear traction. Deformations in this figure are scaled up 100 times.**

Finally, the nodal points were moved randomly. The greatest of these movements was about 20 percent of the distance to the nearest point in the uniform node arrangement. Figure 11 and 12 show these integration points and their corresponding displacement results respectively.

**Fig. 11. Representation of integration points and nodal points with a random distribution**

To improve the results, the number of integration points is increased and laid between the nodal points. The regular distribution of the integration points which is used here is as shown in Figure 15. The nodal points after the final load are plotted in Figure 16. As can be seen, the fluctuations are completely eliminated.
To investigate the computational cost of the present method, the evaluation of the plate under linear traction shown in Figure 13 have been compared with FEM. In this comparison, in the presented method, nodal and integration points have been regularly distributed as shown in Figure 15. Our own FEM program has been used for this purpose with a 2×2 Gauss integration points and the same number of nodal points. The calculation has been performed for models with 66 and 231 nodal points and has been compared with FEM. The times for each individual computational step are listed separately in Table 3. As it can be seen in Table 3, the presented method is more time consuming than FEM. It is reminded that most of the meshless methods in linear problems are usually time consuming as compared with FEM.

Table 3. Comparison of the computational times for evaluating the results of the plate under linear traction

<table>
<thead>
<tr>
<th>Step</th>
<th>Procedure</th>
<th>66 nodal points</th>
<th>231 nodal points</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FEM</td>
<td>Present method</td>
<td>FEM</td>
</tr>
<tr>
<td>1</td>
<td>Nodes and integration points generation</td>
<td>---</td>
<td>0.033</td>
</tr>
<tr>
<td>2</td>
<td>Nodes and element generation</td>
<td>0.0577</td>
<td>---</td>
</tr>
<tr>
<td>3</td>
<td>Calculation of stiffness matrix</td>
<td>0.082</td>
<td>0.249</td>
</tr>
<tr>
<td>4</td>
<td>Calculation of the results</td>
<td>0.061</td>
<td>0.142</td>
</tr>
<tr>
<td></td>
<td>Sum.</td>
<td>0.200</td>
<td>0.424</td>
</tr>
</tbody>
</table>

To validate the results the normal stress in the x direction along the line x=0 is presented and compared with the exact solution given by Eq. 24 (Timoshenko & Goodier, 1970):

\[
\sigma_x(x, y) = 1 - \frac{\alpha^2}{\pi^2} \left\{ \frac{3}{2} \cos 2\theta + \cos 4\theta \right\} + \frac{3\alpha^4}{2\pi^4} \cos 4\theta
\]

(25)

5.3 Plate with a circular hole

Finally, the numerical results of an infinite plate with a circular hole under uniaxial tension are presented. Consider a plate with a circular hole with the radius \(a=1\,\text{m}\). The plate is under a uniform tension; of \(\sigma_{xx}=10^7\,\text{MPa}\) in the x direction, as shown in Figure 18.

Due to symmetry, only a quarter of the upper right quadrant of the plate is modeled, as shown in Figure 18, and its traction and symmetric boundary conditions are imposed. In order to compare the results with the exact solution, the minimum length of the plate should be more than 9a. In this case, the nodal points and integration points are the same and their arrangement is shown in Figure 19.

To validate the results the normal stress in the x direction along the line x=0 is presented and compared with the exact solution given by Eq. 24 (Timoshenko & Goodier, 1970):
where \((r, \theta)\) are the polar coordinates and \(\theta\) is measured counterclockwise from the positive \(x\) axis. The results are presented in Figure 20(a) and the relative error compared with the exact solution is plotted in Figure 20(b).

The maximum calculated concentration factors are shown in Table 4 as a function of number of nodes.

**Fig. 18.** An infinite plate with a circular hole and its upper right quadrant

**Fig. 19.** Distribution of employed 408 nodal points for the plate with a circular hole

**Fig. 20(a)** The \(\sigma_{xx}\) stress component in the horizontal direction along the line with 408 integration and nodal points, (b) relative error of \(\sigma_{xx}\) stress component compared with the exact solution.

### 6. Conclusions

In this paper, the Kriging interpolation method is used for obtaining the weights of the integration points in meshless methods based on global weak form. In the presented method the background mesh is omitted and it was shown that a few number of integration points can be used. Also, the arrangement of the integration points is independent from the nodal points. In several examples, the results of the presented method were compared with the exact solution or the FEM results. The numerical examples presented in this paper demonstrated that the number of the integration points can be the same as that of the nodal points.

The computational cost of the present method has been compared with FEM and showed that the presented method has a reasonable computational efficiency.

In the example of the plate with a circular hole, it was shown that the method is converging toward the exact solution. Improved results were obtained by raising the number of integration points.

Based on the obtained results, it is believed that the presented method is very encouraging and can be applied successfully to evaluate the integrals in meshless methods using global weak formulations.

### 7. References

8. T. Belytschko L. Gu and YY. Lu (1994) Fracture and crack growth by element-free Galerkin methods Model Simu-


