Folding And Fundamental Groups of Buchdahi Space

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Abstract

The purpose of this paper is to give a combinatorial characterization and also construct representations of the fundamental group of the submanifolds of Buchdahi space by using some geometrical transformations. The homotopy groups of the limit Buchdahi space are presented. The fundamental groups of some types of geodesics in Buchdahi space are discussed. New types of homotopy maps are deduced. Theorems governing this connection are achieved.

Keywords: Buchdahi space; Homotopy groups; Folding; Deformation retracts.


1. Introduction and definitions

Buchdahi space represents one of the most intriguing and emblematic discoveries in the history of geometry. Although if it were introduced for a purely geometrical purpose, they came into prominence in many branches of mathematics and physics. This association with applied science and geometry generated synergistic effect: applied science gave relevance to Buchdahi space and Buchdahi space allowed formalizing practical problems.

In vector spaces and linear maps; topological spaces and continuous maps; groups and homomorphisms together with the distinguished family of maps is referred to as a category. An operator which assigns to every object in one category a corresponding object in another category and to every map in the first a map in the second in such a way that compositions are preserved and the identity map is taken to the identity map is called a functor. Thus, we may summarize our activities thus far by saying that we have constructed a functor (the fundamental group functor) from the category of pointed spaces and maps to the category of groups and homomorphisms. Such functors are the vehicles by which one translates topological problems into algebraic problem [17, 18, 19, 20].

Most folding problems are attractive from a pure mathematical standpoint, for the beauty of the problems themselves. The folding problems have close connections to important industrial applications Linkage folding has applications in robotics and hydraulic tube bending. Paper folding has application in sheet-metal bending, packaging, and air –bag folding [12, 14]. Following the great Soviet geometer [11], also, used folding to solve difficult problems related to shell structures in civil engineering and aero space design, namely buckling instability [10]. Isometric folding between two Riemannian manifold may be characterized as maps that send piecewise geodesic segments to a piecewise geodesic segments of the same length [4]. For a topological folding the maps do not preserves lengths[5,6]. i.e. A map \( \Psi: M \to N \), where M and N are \( C^\infty \)-Riemannian manifolds of dimension \( m, n \) respectively is said to be an isometric folding of M into N, iff for any piecewise geodesic path \( y: J \to M \), the induced path \( \Psi \circ y: J \to N \) is a piecewise geodesic and of the same length as \( y \). If \( \Psi \) does not preserve length, then \( \Psi \) is a topological folding [1, 2].

A subset \( A \) of a topological space \( X \) is called a retract of \( X \) if there exists a continuous map \( r: X \to A \) such that \( r(a) = a \), \( \forall a \in A \) where \( A \) is closed and \( X \) is open [3,7]. Also, let \( X \) be a space and \( A \) a subspace. A map \( r: X \to A \) such that \( r(a) = a \), for all \( a \in A \), is called a retraction of \( X \) onto \( A \) and \( A \) is the called a retract of \( X \). This can be re stated as follows. If \( i: A \to X \) is the inclusion map, then \( r: X \to A \) is a map such that \( ri = id_A \). If, in addition, \( ri = id_X \), we call \( r \) a deformation retract and \( A \) a deformation retract of \( X \). Another simple-but extremely useful-idea is that of a retract. If \( A, X \subseteq M \), then \( A \) is a retract of \( X \) if there is a commutative diagram.

\[
\begin{array}{ccc}
A & \xrightarrow{i} & X \\
\downarrow{id_A} & & \downarrow{r} \\
A & \xrightarrow{r} & A
\end{array}
\]
If $f : A \rightarrow B$ and $g : X \rightarrow Y$, then $f$ is a retract of $g$ if there is a commutative diagram [8, 9, 15, 16, 21, 22, 23, 24].

\[ \begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{id_B} \\
X & \xrightarrow{\text{id}_A} & Y
\end{array} \]

### 2. Main results

**Theorem 1.** The fundamental group of types of the deformation retracts of Buchdahi space $\mathcal{B}^4$ are isomorphic to identity group.

**Proof.** Consider the Buchdahi space $\mathcal{B}^4[6,13]$. Used cylindrical coordinates $z, r, \theta$, and $t$ with metric, $ds^2 = \gamma^2 \left( -dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, dt^2 \right) + p^{-1} \, dt^2 \quad (1)$ and take $\gamma = \gamma (t)$.

The coordinates of Buchdahi space $\mathcal{B}^4$ are given by

\[
\begin{align*}
x_1 &= \frac{A_1}{1-\sqrt{p}} \sin \left( \frac{A_2}{1-\sqrt{p}} \right) \cos \left( \frac{A_3}{1-\sqrt{p}} \right) \\
x_2 &= \frac{A_1}{1-\sqrt{p}} \sin \left( \frac{A_2}{1-\sqrt{p}} \right) \sin \left( \frac{A_3}{1-\sqrt{p}} \right) \\
x_3 &= \frac{A_1}{1-\sqrt{p}} \cos \left( \frac{A_2}{1-\sqrt{p}} \right) \\
x_4 &= \frac{A_4}{1-\sqrt{p}}
\end{align*}
\]

Where $A_1, A_2, A_3, \text{and } A_4$ are the constant of integration.

Now, we use Lagrangian equations

\[
\frac{d}{ds} \left( \frac{\partial \mathcal{T}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{T}}{\partial q_i} = 0, \quad i = 1, 2, 3, 4
\]

To find a geodesic which is a subset of the Buchdahi space $\mathcal{B}^4$. Since

\[
T = \frac{1}{2} \left\{ -\gamma^2 \left( -r^2 + r^2 \theta^2 + r^2 \sin^2 \theta \phi^2 \right) + p^{-2} t^2 \right\}.
\]

Then the Lagrangian equations for Buchdahi space $\mathcal{B}^4$ are

\[
\frac{d}{ds} \left( \gamma^2 r^2 \phi \right) + \left( -\gamma^2 r^2 \phi \right) = 0 \quad (3)
\]

\[
\frac{d}{ds} \left( \gamma^2 r^2 \theta \right) + \left( \gamma^2 r^2 \sin \theta \phi \right) = 0 \quad (4)
\]

\[
\frac{d}{ds} \left( \gamma^2 r^2 \sin^2 \theta \phi \right) = 0 \quad (5)
\]

\[
\frac{d}{ds} (p^{-1} t) = 0 \quad (6)
\]

From equation (5) we obtain $\gamma^2 r^2 \sin^2 \theta \phi = \text{constant} \quad \text{say } \beta_1$, if $\beta_1 = 0$, we obtain the following cases: If $\gamma^2 = 0$, then the coordinates of Buchdahi space $\mathcal{B}^4$ are given by

\[
(A_1 \sin A_2 \cos A_3, A_1 \sin A_2 \cos A_3, A_1 \cos A_2, \frac{A_4}{1-\sqrt{p}}).
\]

Which is a hypersphere $S^3 \subset \mathcal{B}^4$, $-x_1^2 - x_2^2 - x_3^2 + x_4^2 = A_1^2 + \left( \frac{A_4}{1-\sqrt{p}} \right)^2$ which is a geodesic retraction. If $r^2 = 0$, hence we get the hypersphere $S^2 \subset \mathcal{B}^4$, $-x_1^2 - x_2^2 - x_3^2 + x_4^2 = 0$, on the null cone which is a geodesic retraction. If $\phi = 0$, then $\phi = \text{constant} \quad \text{say } \beta_2$, if $\beta_2 = 0$, then we obtain the following geodesic retraction

\[
S^2 \subset \mathcal{B}^4 \text{ given by } \left( \frac{A_1}{1-\sqrt{p}} \sin \left( \frac{A_2}{1-\sqrt{p}} \right), 0, \frac{A_3}{1-\sqrt{p}} \cos \left( \frac{A_2}{1-\sqrt{p}} \right), \frac{A_4}{1-\sqrt{p}} \right),
\]

where, $-x_1^2 - x_2^2 - x_3^2 + x_4^2 = \left( \frac{A_1}{1-\sqrt{p}} \right)^2 + \left( \frac{A_4}{1-\sqrt{p}} \right)^2$.

The deformation retract of the Buchdahi space $\mathcal{B}^4$ is defined by $\eta : \mathcal{B}^4 - (\mu_i) \times I \rightarrow \{ \mathcal{B}^4 - (\mu_i) \}$, where $\{ \mathcal{B}^4 - (\mu_i) \}$ is the open Buchdahi space $\mathcal{B}^4$ and $I$ is the closed interval $[0, 1]$. The retraction of the open Buchdahi space $\mathcal{B}^4$ is $R : \{ \mathcal{B}^4 - (\mu_i) \} \rightarrow S^3, S^2, S^1$.

The deformation retract of $S^3 \subset \mathcal{B}^4$ is given by

\[
\eta(m, c) = \cos \left( \frac{nc}{2} \right) \left\{ \frac{A_1}{1-\sqrt{p}} \sin \left( \frac{A_2}{1-\sqrt{p}} \right) \cos \left( \frac{A_3}{1-\sqrt{p}} \right), \frac{A_1}{1-\sqrt{p}} \sin \left( \frac{A_2}{1-\sqrt{p}} \right) \sin \left( \frac{A_3}{1-\sqrt{p}} \right) \cos \left( \frac{A_2}{1-\sqrt{p}} \right), \frac{A_1}{1-\sqrt{p}} \cos \left( \frac{A_2}{1-\sqrt{p}} \right), \frac{A_4}{1-\sqrt{p}} \right\}
\]

\[
\sim \left( \frac{A_1}{1-\sqrt{p}} \sin \left( \frac{A_2}{1-\sqrt{p}} \right) \cos \left( \frac{A_3}{1-\sqrt{p}} \right), \frac{A_1}{1-\sqrt{p}} \sin \left( \frac{A_2}{1-\sqrt{p}} \right) \sin \left( \frac{A_3}{1-\sqrt{p}} \right) \cos \left( \frac{A_2}{1-\sqrt{p}} \right), \frac{A_1}{1-\sqrt{p}} \cos \left( \frac{A_2}{1-\sqrt{p}} \right), \frac{A_4}{1-\sqrt{p}} \right) - (\mu_i)
\]

\[
\eta(m, 1) = \{ A_1 \sin A_2 \cos A_3, A_1 \sin A_2 \sin A_3, A_1 \cos A_2, \frac{A_4}{1-\sqrt{p}} \}.
\]

The deformation retract of $S^2 \subset \mathcal{B}^4$ is defined as

\[
\eta(m, c) = \frac{1-c}{1+c} \left\{ \frac{A_1}{1-\sqrt{p}} \sin \left( \frac{A_2}{1-\sqrt{p}} \right) \cos \left( \frac{A_3}{1-\sqrt{p}} \right), \frac{A_1}{1-\sqrt{p}} \sin \left( \frac{A_2}{1-\sqrt{p}} \right) \sin \left( \frac{A_3}{1-\sqrt{p}} \right) \cos \left( \frac{A_2}{1-\sqrt{p}} \right), \frac{A_1}{1-\sqrt{p}} \cos \left( \frac{A_2}{1-\sqrt{p}} \right), \frac{A_4}{1-\sqrt{p}} \right\} - (\mu_i)
\]

\[
+ c(2c - 1) \{ 0, 0, 0, 0 \}.
\]
The deformation retract of $S_3^3 \subset B^4$ is defined as
\[
\eta(m, e) = \ln e^{(1-C)} \{ A_1 \frac{A_2}{1-i\sqrt{p}} \sin \left( \frac{A_3}{1-i\sqrt{p}} \sin \phi \right), A_1 \frac{A_2}{1-i\sqrt{p}} \} + \ln e^{(C)} \{ A_1 \frac{A_2}{1-i\sqrt{p}} \sin \left( \frac{A_3}{1-i\sqrt{p}} \sin \phi \right), A_1 \frac{A_2}{1-i\sqrt{p}} \sin \left( \frac{A_3}{1-i\sqrt{p}} \sin \phi \right) - (\mu_i) \}.
\]

**Corollary 1.** The fundamental group of types of the deformation retracts of $S_3^3 \subset B^4$ and any manifold homeomorphic Buchdahi space $B^4$ is isomorphic to the identity group.

**Theorem 2.** The fundamental group of the limit limit of foldings of the hypersphere $S_3^3 \subset B^4$ and any manifold homeomorphic to it is isomorphic to $\mathbb{Z}$.

Proof. Consider the hypersphere $S_3^3$ and let $\mathscr{X}_1 : S_3^3 \to S_3^3$, be a folding map, now we can define a series of folding maps by
\[
\mathcal{X}_2 : \mathcal{X}_1(S_3^3) \to \mathcal{X}_1(S_3^3),
\]
\[
\mathcal{X}_3 : (\mathcal{X}_1(S_3^3)) \to \mathcal{X}_2(S_3^3),
\]
\[
\mathcal{X}_n : \mathcal{X}_{n-1}(S_3^3) \to \mathcal{X}_{n-1}(S_3^3),
\]
\[
\lim_{n \to \infty} \mathcal{X}_n(S_3^3) = \text{is a great sphere } (S_3^3) \text{ of dimension two. Therefore } \pi_1(S_3^3) \text{ is isomorphic to identity group.}
\]

If, we consider
\[
\gamma_1 : S_3^2 \to S_3^2, \gamma_2 : \gamma_1(S_3^2) \to \gamma_1(S_3^2),
\]
\[
\gamma_3 : \gamma_2(\gamma_1(S_3^2)) \to \gamma_2(\gamma_1(S_3^2)),
\]
\[
\gamma_n : \gamma_{n-1}(\gamma_{n-2}(\gamma_{n-3}(\gamma_{n-4}(\gamma_{n-5}(\gamma_2(S_3^2)))))) \to \gamma_{n-1}(\gamma_{n-2}(\gamma_{n-3}(\gamma_{n-4}(\gamma_{n-5}(\gamma_2(S_3^2)))))),
\]
\[
\lim_{n \to \infty} \gamma_n(S_3^2) = (S_3^3) \subset B^4 \text{ which is the great circle of dimension one. Therefore } \pi_1(S_3^3) \subset B^4 \text{ is isomorphic to } \mathbb{Z}.
\]

**Theorem 3.** Under the folding $\Pi_m : (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, \frac{x_4}{m})$ the fundamental group of the limit of foldings and any manifold homeomorphic to this type of folding of $S_3^3$ is isomorphic to $\mathbb{Z}$.

Thus, $\pi_1(M - (\mu)) \approx \pi_1(S_3^3)$, $\pi_1(M - (\mu)) \approx \pi_1(S_3^3)$, and $\pi_1(M - (\mu)) \approx \pi_1(S_3^3)$.

Proof. Consider the geodesic retraction hypersphere $S_3^3$ and let $\Pi_m : S_3^3 \to S_3^3$ be given by $\Pi_m (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, \frac{x_4}{m})$.

Then, the isometric chain folding of $S_3^3$ into itself defined by
\[
\Pi_1 : \frac{A_1}{1-i\sqrt{q}} \sin \left( \frac{A_2}{1-i\sqrt{q}} \right), 0, \frac{A_1}{1-i\sqrt{q}} \cos \left( \frac{A_2}{1-i\sqrt{q}} \right), \frac{A_4}{1-i\sqrt{q}} + \Pi_2 : \frac{A_1}{1-i\sqrt{q}} \sin \left( \frac{A_2}{1-i\sqrt{q}} \right), 0, \frac{A_1}{1-i\sqrt{q}} \cos \left( \frac{A_2}{1-i\sqrt{q}} \right), \frac{A_4}{1-i\sqrt{q}} + \Pi_m : \frac{A_1}{1-i\sqrt{q}} \sin \left( \frac{A_2}{1-i\sqrt{q}} \right), 0, \frac{A_1}{1-i\sqrt{q}} \cos \left( \frac{A_2}{1-i\sqrt{q}} \right), \frac{A_4}{1-i\sqrt{q}} \]

Then we get
\[
\lim_{m \to \infty} \Pi_m = \frac{A_1}{1-i\sqrt{q}} \sin \left( \frac{A_2}{1-i\sqrt{q}} \right), 0 \frac{A_1}{1-i\sqrt{q}} \cos \left( \frac{A_2}{1-i\sqrt{q}} \right), \frac{A_4}{1-i\sqrt{q}} \]

Thus, $\frac{A_1}{1-i\sqrt{q}} = \frac{x_4}{m}$, which is the geodesic retraction great circle $S_3^3 \subset S_3^3$ with $x_2 = x_4 = 0$. Therefore, $\pi_1(S_3^3) \subset S_3^3$, is isomorphic to $\mathbb{Z}$.

**Corollary 2.** Under the folding $\Pi_m : (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, \frac{x_4}{m})$ the fundamental group of the limit of foldings of the hypersphere $S_3^3$, is a hypersphere $S_3^3 \subset S_3^3$ which is isomorphic to the identity group.

**Corollary 3.** Under the folding $\Pi_m : (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, \frac{x_4}{m})$ and $A_3 = 0$, the fundamental group of the limit of foldings of the hypersphere $S_3^3$, is a great circle $S_3^3 \subset S_3^3$, which is isomorphic to $\mathbb{Z}$.

**Corollary 4.** Under the folding $\Pi_m : (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, \frac{x_4}{m})$ and $A_3 = 0$, the fundamental group of the
deformation retract of \( S^1 \) onto the great circle \( S^2_3 \subset S^1 \) is isomorphic to \( Z \).

**Theorem 4.** Under the folding \( \prod_m (x_1, x_2, x_3, x_4) = (\frac{x_1}{m}), \frac{x_2}{m}, \frac{x_3}{m}, \frac{x_4}{m} \) the fundamental group of the limit of foldings of the Buchdahi space \( B^4 \) is isomorphic to the identity group.

Proof. Now consider the Buchdahi space \( B^4 \) of dimension four and let

\[ \prod_m : B^4 \rightarrow B^4 \] be given by \( \prod_m (x_1, x_2, x_3, x_4) = (\frac{x_1}{m}), \frac{x_2}{m}, \frac{x_3}{m}, \frac{x_4}{m} \)

Then, the isometric chain folding of the Buchdahi space \( B^4 \) into itself may be defined by

\[
\begin{align*}
\Pi_1 : & \begin{cases}
A_1 \left[ \frac{A_2}{1-i\sqrt{p}} \sin \left( \frac{A_3}{1-i\sqrt{p}} \right) \cos \left( \frac{A_4}{1-i\sqrt{p}} \right) \right], \\
A_1 \left[ \frac{A_2}{1-i\sqrt{p}} \sin \left( \frac{A_3}{1-i\sqrt{p}} \right) \cos \left( \frac{A_4}{1-i\sqrt{p}} \right) \right], \\
A_1 \left[ \frac{A_2}{1-i\sqrt{p}} \sin \left( \frac{A_3}{1-i\sqrt{p}} \right) \cos \left( \frac{A_4}{1-i\sqrt{p}} \right) \right], \\
A_1 \left[ \frac{A_2}{1-i\sqrt{p}} \sin \left( \frac{A_3}{1-i\sqrt{p}} \right) \cos \left( \frac{A_4}{1-i\sqrt{p}} \right) \right], \\
A_1 \left[ \frac{A_2}{1-i\sqrt{p}} \sin \left( \frac{A_3}{1-i\sqrt{p}} \right) \cos \left( \frac{A_4}{1-i\sqrt{p}} \right) \right],
\end{cases}
\end{align*}
\]

Then, the isometric chain folding of the Buchdahi space \( B^4 \) into itself may be defined by

\[
\begin{align*}
\Pi_2 : & \begin{cases}
A_1 \left[ \frac{A_2}{1-i\sqrt{p}} \sin \left( \frac{A_3}{1-i\sqrt{p}} \right) \cos \left( \frac{A_4}{1-i\sqrt{p}} \right) \right], \\
A_1 \left[ \frac{A_2}{1-i\sqrt{p}} \sin \left( \frac{A_3}{1-i\sqrt{p}} \right) \cos \left( \frac{A_4}{1-i\sqrt{p}} \right) \right], \\
A_1 \left[ \frac{A_2}{1-i\sqrt{p}} \sin \left( \frac{A_3}{1-i\sqrt{p}} \right) \cos \left( \frac{A_4}{1-i\sqrt{p}} \right) \right], \\
A_1 \left[ \frac{A_2}{1-i\sqrt{p}} \sin \left( \frac{A_3}{1-i\sqrt{p}} \right) \cos \left( \frac{A_4}{1-i\sqrt{p}} \right) \right], \\
A_1 \left[ \frac{A_2}{1-i\sqrt{p}} \sin \left( \frac{A_3}{1-i\sqrt{p}} \right) \cos \left( \frac{A_4}{1-i\sqrt{p}} \right) \right],
\end{cases}
\end{align*}
\]

Thus, \( x_1^2 - x_2^2 - x_3^2 + x_4^2 = \left( \frac{A_1}{1-i\sqrt{p}} \cos \left( \frac{A_2}{1-i\sqrt{p}} \right) \right)^2 + \left( \frac{A_4}{1-i\sqrt{p}} \right)^2 \), which is hypersurface \( B_1 \subset B^4 \) with \( x_1 = x_2 = 0 \); Therefore \( \pi_1 (B_1 \subset B^4) \) is isomorphic to the identity group.

**Theorem 5.** Under the folding \( \prod_m (x_1, x_2, x_3, x_4) = (\frac{x_1}{m}), \frac{x_2}{m}, \frac{x_3}{m}, \frac{x_4}{m} \) the fundamental group of the limit of foldings of the Buchdahi space \( B^4 \) is the identity group.

Proof. Consider the four dimension Buchdahi space \( B^4 \) and let

\[ \prod_m : B^4 \rightarrow B^4 \] be given by \( \prod_m (x_1, x_2, x_3, x_4) = (\frac{x_1}{m}), \frac{x_2}{m}, \frac{x_3}{m}, \frac{x_4}{m} \)

Then, the isometric chain folding of the Buchdahi space \( B^4 \) into itself get

\[ \text{lim}_{m \rightarrow \infty} \prod_m = \{0,0,0,0\} \] which a zero- dimensional Buchdahi space \( B^0 \). Thus, it is a point and the fundamental group of a point is the identity group.

**Corollary 5.** The fundamental group of the end limits of foldings of the n-dimensional manifold \( F^n \) homeomorphic to \( n \)-dimensional Buchdahi space \( B^n \) into itself is the identity group.

**Theorem 6.** The fundamental group of the minimal retraction of the \( n \)-dimensional manifold \( F^n \) homeomorphic to \( n \)-dimensional Buchdahi space \( B^n \) is the identity group.

Proof. Let \( r_1 : \{ F^n - (\beta_1^n) \} \rightarrow F^{n-1} \) be the retraction map. Then, we have the following chains

\[
\begin{align*}
r_1 & = \{ F^n - (\beta_1^n) \} \rightarrow \{ F^n - (\beta_2^n) \} \rightarrow \{ F^n - (\beta_3^n) \} \rightarrow \cdots \} \\
& \{ F^n - (\beta_n^n - 1) \} \rightarrow F^{n-1},
\end{align*}
\]

\[
\begin{align*}
r_2 & = \{ F^n - (\beta_1^n) \} \rightarrow \{ F^n - (\beta_2^n) \} \rightarrow \{ F^n - (\beta_3^n) \} \rightarrow \cdots \} \\
& \{ F^n - (\beta_n^n - 1) \} \rightarrow F^{n-2},
\end{align*}
\]

Thus from the above chain the minimal retractions of the \( n \)-dimensional manifold \( F^n \) coincides with the zero-dimensional space which is the limit of retractions. Thus, it is a point and the fundamental group of a point is the identity group.
**Theorem 7.** The fundamental group of the retraction of Buchdahi plane $B^2$ is isomorphic to $\mathbb{Z}$.

**Proof.** Since $\gamma = \frac{1}{2} \ln BC$ [1], if $B = C$, then $\gamma = \ln B$. When $\ln B = 1$, implies $\gamma = 1$. Hence (1) becomes $s^2 = - (dr^2 + r^2 d\theta^2 + 2r^2 \sin^2 \theta \, d\phi^2) + p^{-1} dt^2$. Also, under the condition $t = 0$, then $ds^2$ implies that $ds^2 = - (dr^2 - r^2 d\theta^2)$. Now $S^2_\gamma$ is a retract of $B^2 - \{(0,0)\}$, where $r: B^2 - \{(0,0)\} \to S^1_\gamma$ defined by $r(x) = \frac{x}{\|x\|}$ is a continuous map. Therefore $\pi_1(S^1_\gamma)$ is isomorphic to $\mathbb{Z}$.

**Theorem 8.** The fundamental group of the deformation retract of the Buchdahi plane $B^2$ is isomorphic to $\mathbb{Z}$.

**Proof.** Since $S^1_\gamma$ is a retract of $B^2 - \{(0,0)\}$, but a subset $A$ of a Buchdahi plane $B^2$ is said to be a deformation retract of the Buchdahi plane $B^2$ if there exists a homotopy map $F : (B^2 - \{(0,0)\}) \times [0,1] \to (B^2 - \{(0,0)\})$ defined as $F(x,t) = (1 - t)x + t \frac{x}{\|x\|}$ such that $F_0 = \text{id}_{B^2}$ and $F_1 : (B^2 - \{(0,0)\}) \to (B^2 - \{(0,0)\})$ satisfies $F_1(x) \in A$ for every $x \in (B^2 - \{(0,0)\})$ and $F_1(S^1_\gamma) = S^1_\gamma$ for every $S^1_\gamma \in A$. Hence $F$ is a deformation retract of the Buchdahi plane onto $S^1_\gamma$ and $\pi_1(S^1_\gamma)$ is isomorphic to $\mathbb{Z}$.

**Theorem 9.** The fundamental group of any types of folding of $S^n \subset B^{n+1}$ such that $\dim S(S^n) = \dim S^n$, $n \geq 2$, is the identity group.

**Proof.** Now, consider the foldings with singularity of $S^n \subset B^{n+1}$ to a subset of $S^n \subset B^{n+1}$ such that $\dim S(S^n) = \dim (S^n)$, then all loops of $S(S^n)$ are homotopic to the identity loop, and hence the fundamental group of this types of folding is the identity group as in Fig. (1). Again, consider the foldings without singularity of $S^n \subset B^{n+1}$ and also the fundamental group of this types of foldings is the identity group as in Fig. (2).

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**Theorem 10.** Let $M \subset B^2$ be the union of the circles $C^n = \bigcup_{(x \in \mathbb{Z} - \{0\})} S^1(\frac{1}{x}, 0, \frac{1}{x}) \subset B^2$. Then there are folding $\nu: M \to M$ and retractions $r: M \to C^n$ such that $\pi_1(n(M)) = \pi_1(r(M))$ and $\pi_1(\nu(M))$ is either identity group or isomorphic to $\mathbb{Z}$.

**Proof.** Let $\nu: M \to M$ be a folding such that $\nu(C^n) = C^n$, then $\nu(M) = C^n$. Also, consider the retractions $r_n: M \to C^n$, which collapsing all $C^n$ except $C^n$ to the origin and too $r_n(M) = C^n$, then $\pi_1(\nu(M)) = \pi_1(r_n(M))$. Now, if $n \to \infty$ then $C^n$ is a point and so $\pi_1(n(M)) = 0$, otherwise if $n \not\to \infty$ then $C^n$ is a circle and $\pi_1(n(M))$ is isomorphic to $\mathbb{Z}$. See Fig. (3).

**Theorem 11.** Let $M \subset B^2$ be the union of the circles $C^n = \bigcup_{(x \in \mathbb{Z} - \{0\})} S^1(\frac{1}{x}, 0, \frac{1}{x}) \subset B^2$. Then there are folding $\nu: M \to M$ which induces a folding $\nu_1(M) \to \pi_1(M)$ such that $\nu_1(M) = \nu_1(M)$ and $\nu_1(M)$ is uncountable.

**Proof.** Let $n: M \to M$ be a folding such that $n(C^n) = n\bigcup_{(x \in \mathbb{Z} - \{0\})} S^1(\frac{1}{x}, 0, \frac{1}{x}) \subset B^2 = C^n$. Also, $n(C^n)$ satisfies $n_1(C^n) = n\bigcup_{(x \in \mathbb{Z} - \{0\})} S^1(\frac{1}{x}, 0, \frac{1}{x}) \subset B^2 = C^n$. Thus we get the induced folding $\nu_1(M) \to \pi_1(M)$ such that $\nu_1(M) = \nu_1(M)$ since, $\pi_1(M)$ is uncountable, it follows that $\nu_1(M)$ is uncountable. See Fig. (4).
Corollary 6. Let $M \subset \mathcal{B}^2$ be the union of the circles $C^n = \bigcup_{z \in (-\infty,0]} S^1 \left( \frac{1}{\sqrt{1+z^2}}, 0 \right) \subset \mathcal{B}^2$. If $\mathcal{F} : M \to M$ be a folding defined as $\mathcal{F}(x, y) = (\|x\|, \|y\|)$. Then $\pi_1(\mathcal{F}(M)) = 0$.

3. Conclusion

In this paper we achieved the approval of the important of the fundamental group in the submanifolds of Buchdahi Space by using some geometrical transformations. The relations between folding, retractions, deformation retract limits of folding and limits of retractions of the fundamental group in the submanifolds of Buchdahi Space are discussed. The connection between limits of folding and the fundamental group are obtained. New types of minimal retractions on the fundamental group are deduced.

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5. References