On some metacyclic $cp$-groups and their nonabelian tensor square

A M Basri, N H Sarmin, N M Mohd Ali and J R Beuerle

Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, Johor, Malaysia
Mathematics and Statistics Department, Elon University, North Carolina, USA

nhs@utm.my, nizasarmin67@yahoo.com

Abstract

In this paper, we develop appropriate programme using Groups, Algorithms and Programming (GAP) software enables performing different computations on various characteristics of all finite nonabelian metacyclic $cp$-groups. $p$ is an odd prime, of nilpotency class at least three. Such programme enables to compute structure of the group, order of the group, structure of the center, the number of conjugacy classes, structure of commutator subgroup, abelianization, Whitehead’s universal quadratic functor and other characteristics. In addition, structures of some other groups such as the nonabelian tensor square and various homological functors including Schur multiplier and exterior square can be computed using this programme. Furthermore, by computing the epicenter order or the exterior center order the capability can be determined. In our current article, we only compute the nonabelian tensor square of certain order groups, as an example, and give GAP codes for computing other characteristics and some subgroups.

Keywords: $p$-Groups; Metacyclic Groups; Groups, Algorithms and Programming

1. Introduction

A group $G$ is metacyclic if it contains a cyclic normal subgroup $N$ such that $G/N$ is also cyclic. Finite metacyclic groups can be presented on two generators and three defining relations. Some specific types of metacyclic groups have been given much attention. Zassenhaus discussed metacyclic groups with cyclic commutator quotient, as did Hall. Beyl, King, Liedahl, Newman and Xu, Rédei and Lindenberg all classified metacyclic $cp$-groups of odd order. Significant progress was made in 1994 when Sim proved in [1] that every finite metacyclic group can be decomposed naturally as a semidirect product of two Hall subgroups [2].

A $p$-group is a group in which each element has a power of $p$ as its order whereas a finite group is a $p$-group if and only if its order is a power of $p$ [3].

By Beuerle [4], a metacyclic $cp$-group is usually given by a presentation of the form:

$$G = \langle a, b; a^{p^m} = 1, b^{p^n} = a^k, bab^{-1} = a^e \rangle,$$

where $m,n \geq 0$, $r > 0$, $k \leq p^r$, $p^m | k(r-1)$ and $p^n | r^p - 1$.

Metacyclic $cp$-groups are divided by Beuerle [4] into two main classes as follows:

$$G_p(\alpha, \beta, \delta, \pm) = \langle a, b; a^{p^m} = 1, b^{p^n} = a^{p^{\alpha+\delta}}, bab^{-1} = a^\delta \rangle,$$

where $r = p^{\alpha-\delta} + 1$ or $r = p^{\alpha-\delta} - 1$. We say that the group is of positive or negative type if $r = p^{\alpha-\delta} + 1$ or $r = p^{\alpha-\delta} - 1$, respectively.

For short, we write $G_+$ and $G_-$ for $G_p(\alpha, \beta, \delta, \pm)$ and $G_p(\alpha, \beta, \delta, -)$ respectively.

The following proposition gives a standardized parametric presentation of metacyclic $cp$-groups.

**Proposition 1:** [4] Let $G$ be a finite metacyclic $cp$-group. Then there exist integers $\alpha, \beta, \delta, \epsilon$ with $\alpha, \beta > 0$ and $\delta, \epsilon$ nonnegative, where $\delta \leq \min\{\alpha - 1, \beta\}$ and $\delta + \epsilon \leq \alpha$ such that for an odd prime $p, G \cong G_p(\alpha, \beta, \delta, \epsilon)$. If $p = 2$, then in addition $\alpha - \delta > 1$ and $G \cong G_2(\alpha, \beta, \delta, +)$ or $G \cong G_p(\alpha, \beta, \delta, -), \delta > 0$. Moreover, if $G$ is of positive type, then $\delta > 0$ for all $p$.

Some of the basic properties and various subgroups are given in the next proposition:

**Proposition 2:** [4] Let $G$ be a group of type $G_p(\alpha, \beta, \delta, \pm)$ and $k \geq 1$. Then we have the following results for the order of $G$, the center of $G$, the order of the center of $G$, the derived subgroups of $G$, the order of the derived subgroups of $G$ and the exponent of $G$:

<table>
<thead>
<tr>
<th>$G_+$</th>
<th>$G_-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>G_+</td>
</tr>
<tr>
<td>$Z(G_+)$</td>
<td>$\langle a^{p^{\alpha+\beta-2\delta}}, b^{p^{\alpha-\delta}}, a^{p^{\alpha+\beta}}, p^{\alpha+\beta-2\delta} \rangle$</td>
</tr>
<tr>
<td>$Z(G_-)$</td>
<td>$\langle a^{p^{\alpha+\beta}}, b^{\max{1, \delta}}, a^{p^{\alpha+\beta}}, 2^{1+\beta-\max{1, \delta}} \rangle$</td>
</tr>
<tr>
<td>$\gamma_{k+1}(G_+)$</td>
<td>$\langle a^{p^{\alpha+\beta}(\delta-\epsilon)}, b^{p^{\alpha+\beta}(\delta-\epsilon)}, a^{p^{\alpha+\beta}(\delta-\epsilon)} \rangle$</td>
</tr>
<tr>
<td>$\gamma_{k+1}(G_-)$</td>
<td>$\langle a^{p^{\alpha+\beta}(\delta-\epsilon)}, b^{p^{\alpha+\beta}(\delta-\epsilon)}, a^{p^{\alpha+\beta}(\delta-\epsilon)}, 2^{\max{0, \delta-\epsilon}} \rangle$</td>
</tr>
<tr>
<td>Exp($G_+$)</td>
<td>$p^{\max{\alpha, \beta+\epsilon}}$</td>
</tr>
</tbody>
</table>

Exp($G_-$)
Moreover, if $\beta > \max \{1, \delta\}$ then $Z(G_\cdot)$ is cyclic if and only if $\epsilon = 1$.

In [4], Beuerle classified all finite nonabelianmetacyclic groups. Using this classification, we develop appropriate programme using Groups, Algorithms and Programming (GAP) software enabling performing different computations on various characteristics of all finite nonabelianmetacyclic-groups. $p$ is an odd prime, of nilpotency at least three. Such programme enables to compute structure of the group, order of the group, structure of the center of the number of conjugacy classes, commutator subgroup, abelianization, Whitehead’s universal quadratic function and other characteristics. In addition, structures of some other groups such as the nonabelian tensor square and various homological functors including Schur multiplier and exterior square can be computed using this programme. Furthermore, by computing the epicenter order or the exterior center order the capability can be determined. In our current article, we only compute the nonabelian tensor square of certain order groups, as an example, and GAP codes for computing other characteristics and some subgroups given.

The nonabelian tensor square is defined as follows:

**Definition 1:** [5] For a group $G$, the nonabelian tensor square, $G \otimes G$, is generated by the symbols $g \otimes h$, where $g, h \in G$, subject to the relations;

$$gg' \otimes h = (g \otimes g')(h \otimes h)$$

and

$$g \otimes hh' = (g \otimes h)(h \otimes h')$$

for all $g, g', h, h' \in G$ where $h = hgh^{-1}$ denotes the conjugate of $g$ by $h$.

GAP is a system for computational discrete algebra, with emphasis on computational group theory. GAP provides a programming language, a library of functions that implement algebraic algorithms written in the GAP language as well as libraries of algebraic objects such as for all non-isomorphic groups up to order 2000. GAP can be a useful pedagogical tool in different ways such as a fancy calculator, a way to provide large or complicated examples, a way for researcher to write simple computer algorithms, a way for producing large amounts of data so that the researcher can formulate a conjecture and a mean for researchers to work in collaboration [6].

2. The Classification of Groups

In this section, the classification of all finite nonabelianmetacyclic groups, $p$ is an odd, of nilpotency class at least three done by Beuerle in [4] is stated.

2.1 $G \cong \langle x, y; x^{p_\alpha} = y^{p_\beta} = 1, [y, x] = x^{p_\epsilon-\delta}\rangle,$

where $\alpha, \beta, \delta \in \mathbb{N}$, $\delta \leq \alpha < 2\delta, \delta \leq \beta, \delta \leq \min \{\alpha - 1, \beta\}$

2.2 $G \cong \langle x, y; x^{p_\alpha} = 1, y^{p_\beta} = x^{p_\epsilon-\delta}, [y, x] = x^{p_\epsilon-\delta}\rangle$

where $\alpha, \beta, \delta, \epsilon \in \mathbb{N}$, $\delta + \epsilon \leq \alpha < 2\delta, \delta \leq \beta, \alpha < \beta + \epsilon, \epsilon \leq \delta \leq \min \{\alpha - 1, \beta\}$

3. GAP Programmes

In this section, we use GAP to develop appropriate coding for computing different characteristics of the groups and the structure of various subgroups for some small order groups of types (2.1) and (2.2). We provide GAP programmes generating general codes to create the finite presentation of the groups and compute the nonabelian tensor square of some small order groups of types (2.1) and (2.2), as examples.

3.1 Type (2.1)

First, we use GAP programme to generate all finite nonabelianmetacyclic groups of type (2.1) and construct some examples.

3.1.1 Generating Type (2.1)

In this section we develop the GAP coding to generate all finite nonabelianmetacyclic groups of type (2.1).

```
Type2_1:=
function(p,l,beta,n)
if n<=l and l<2*n and m>=n and n<=Minimum(l-1,m) and n>=1 then
  F:=FreeGroup(2);
a:=F.1;b:=F.2;
G:=F/[a^p,b^q,(b*a*b)^(-1)*a^p(l-n+1)];
G:=PcGroupToPcpGroup(G);
ts:=NonAbelianTensorSquare(G);
ATS(G)""); StructureDescription(ts,"\n");
fi;
od;
end;
```

We defined the group with four parameters which are $p$, $\alpha$, $\beta$ and $\delta$. The GAP code below is used to create the finite presentation of the group.

```
F:=FreeGroup(2);
a:=F.1;b:=F.2;
G:=F/[a^p,b^q,(b*a*b)^(-1)*a^p(l-n+1)];
```

3.1.2 Constructing Examples of Type (2.1)

Now, we define the parameters and put them in GAP. For example, we define \( p = 3 \), \( \alpha = 3 \), \( \beta = 2, 3, 4, 5 \) and \( \delta = 2 \). First we need to read the file and call the function (here \( \beta = 2, 3, 4, 5 \)).

\[
gap> \text{Type2}_1(3,3,4,2); \\
p=3 \text{ alpha}=3 \text{ beta}=2 \text{ delta}=2 \text{ NATS(G)}: C_9 \times C_9 \times C_3 \times C_3 \\
p=3 \text{ alpha}=3 \text{ beta}=3 \text{ delta}=2 \text{ NATS(G)}: C_{27} \times C_7 \times C_3 \times C_3 \\
p=3 \text{ alpha}=3 \text{ beta}=4 \text{ delta}=2 \text{ NATS(G)}: C_{81} \times C_7 \times C_3 \times C_3 \\
p=3 \text{ alpha}=3 \text{ beta}=5 \text{ delta}=2 \text{ NATS(G)}: C_{243} \times C_7 \times C_3 \times C_3 \\
\]

Using the coding as before, we get the following more results:

\[
gap> \text{Type2}_1(3,4,4,3); \\
p=3 \text{ alpha}=4 \text{ beta}=3 \text{ delta}=3 \text{ NATS(G)}: C_{27} \times C_9 \times C_3 \times C_3 \\
p=3 \text{ alpha}=4 \text{ beta}=4 \text{ delta}=3 \text{ NATS(G)}: C_{81} \times C_9 \times C_3 \times C_3 \\
gap> \text{Type2}_1(3,5,3,3); \\
p=3 \text{ alpha}=5 \text{ beta}=3 \text{ delta}=3 \text{ NATS(G)}: C_{27} \times C_9 \times C_9 \times C_9 \\
p=5 \text{ alpha}=3 \text{ beta}=2 \text{ delta}=2 \text{ NATS(G)}: C_{25} \times C_5 \times C_5 \times C_5 \\
\]

With similar to the above, we can get much more results for different primes, \( p \), and different values of the parameters \( \alpha, \beta, \delta \).

3.1.3 Summary of GAP Results

A summary of GAP results for type (2.1) is given in the following table:

**Table 1. GAP results for type (2.1)**

<table>
<thead>
<tr>
<th>#</th>
<th>( p )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \delta )</th>
<th>( G \otimes G )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>( C_9 \times C_9 \times C_3 \times C_3 )</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>( C_{27} \times C_7 \times C_3 \times C_3 )</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>( C_{81} \times C_7 \times C_3 \times C_3 )</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>( C_{243} \times C_7 \times C_3 \times C_3 )</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>( C_{27} \times C_7 \times C_3 \times C_3 )</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>( C_{81} \times C_7 \times C_3 \times C_3 )</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>( C_{27} \times C_7 \times C_9 \times C_9 )</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>( C_{25} \times C_5 \times C_5 \times C_5 )</td>
</tr>
</tbody>
</table>

**Remark 1:** Table 1 showed that our numerical results produced by GAP have verified the following theorem given by Beuerle [7]:

**Theorem 1:** Let \( G \) be a group of type \( G_p(\alpha, \beta, 0, \delta, +) \), \( p \) is an odd prime, of class at least three. Then,

\[
G \otimes G \cong C_{p^{\alpha-\delta}} \times C_{p^{\min\{\alpha, \beta\}}} \times C_{p^{\min\{\alpha-\delta, \beta\}}} \times C_{p^\beta}.
\]

3.2 Type (2.2)

With similar objective as for groups of type (2.1), we produce GAP algorithms for groups of type (2.2).

3.2.1 Generating Type (2.2)

In this section we develop the GAP coding to generate all finite nonabelianmetacyclic\( p \)-groups of type (2.2).

\[
\text{Type2}_2:= \\
\text{function}(p,l,beta,n,s) \\
\] localF,m,a,b,G,ts; \\
for m in [1..beta] do \\
if n+s<=l and l<2*n and m>=n and n>=1 and s>=1 and l<m+s and n<=Minimum(l-1,m) then \\
F:=FreeGroup(2); \\
a:=F.1;b:=F.2; \\
G:=F/[a^(p^l),b^(p^m)*a^((-p)^(l-s)),(b*a*b^(-1)*a^(-1))^(-1)*a^(p^(l-n))]; \\
G:=PcGroupToPcpGroup(G); \\
ts:=NonAbelianTensorSquare(G); \\
Print("p=",p," alpha=",l," beta=",m," epsilon=",s," delta=",n" NATS(G):",StructureDescription(ts),"\n"); \\
f1; \\
end;
\]

We defined the group with four parameters which are \( p, \alpha, \beta, \epsilon \) and \( \delta \). The GAP code below is used to create the finite presentation of the group.

\[
F:=\text{FreeGroup}(2); \\
a:=F.1;b:=F.2; \\
G:=F/[a^(p^l),b^(p^m)*a^((-p)^(l-s)),(b*a*b^(-1)*a^(-1))^(-1)*a^(p^(l-n))]; \\
\]

3.2.2 Constructing Examples of Type (2.2)

Now, we define the parameters and put them in GAP. For example, we define, \( p = 3 \), \( \beta = 2, 3, 4, 5, 6 \), \( \epsilon = 1 \) and \( \delta = 2 \).

First we need to read the file and call the function (here \( \beta = 2, 3, 4, 5, 6 \)).

\[
gap> \text{Type2}_1(3,3,7,1,2); \\
p=3 \text{ alpha}=3 \text{ beta}=3 \text{ epsilon}=1 \text{ delta}=2 \text{ NATS(G)}: C_{27} \times C_9 \times C_3 \times C_3 \\
p=3 \text{ alpha}=3 \text{ beta}=4 \text{ epsilon}=1 \text{ delta}=2 \text{ NATS(G)}: C_{81} \times C_9 \times C_3 \times C_3 \\
p=3 \text{ alpha}=3 \text{ beta}=5 \text{ epsilon}=1 \text{ delta}=2 \text{ NATS(G)}: C_{243} \times C_9 \times C_3 \times C_3 \\
p=3 \text{ alpha}=3 \text{ beta}=6 \text{ epsilon}=1 \text{ delta}=2 \text{ NATS(G)}: C_{729} \times C_9 \times C_3 \times C_3 \\
\]
3.2.3 Summary of GAP Results

A summary of GAP results for type (2.1) is given in the following table:

<table>
<thead>
<tr>
<th>#</th>
<th>p</th>
<th>α</th>
<th>β</th>
<th>ε</th>
<th>δ</th>
<th>G ⊗ G</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>C27 × C9 × C3 × C3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>C81 × C9 × C3 × C3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>C243 × C9 × C3 × C3</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>C729 × C9 × C3 × C3</td>
</tr>
</tbody>
</table>

Remark 2: Table 2 showed that our numerical results produced by GAP have verified the following theorem given by Beuerle [7]:

**Theorem 2:** Let \( G \) be a group of type \( G_p(\alpha, \beta, \epsilon, \delta, +) \), \( p \) is an odd prime, of class at least three. Then,

\[
G \otimes G \cong C^p_{\alpha-\delta} \times C^p_{\min\{\alpha-\epsilon, \beta\}} \times C^p_{\min\{\alpha-\beta, \delta\}} \times C^p_{\alpha+\beta}
\]

Remark 3: Given below are GAP codes for computing certain characteristics and subgroups.

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>GAP Codes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order of group</td>
<td>Order(G)</td>
</tr>
<tr>
<td>Center of group</td>
<td>Centre(G)</td>
</tr>
<tr>
<td>Nilpotency class</td>
<td>NilpotencyClassOfGroup(G)</td>
</tr>
<tr>
<td>Commutator subgroup</td>
<td>DerivedSubgroup(G)</td>
</tr>
<tr>
<td>Whitehead`s universal quadratic functor</td>
<td>WhiteheadQuadraticFunctor(G)</td>
</tr>
<tr>
<td>Nonabelian tensor square</td>
<td>NonAbelianTensorSquare(G)</td>
</tr>
<tr>
<td>Schur multiplier</td>
<td>SchurMultiplier(G)</td>
</tr>
<tr>
<td>Exterior square</td>
<td>NonAbelianExteriorSquare(G)</td>
</tr>
<tr>
<td>Epicenter</td>
<td>Epicenter(G)</td>
</tr>
<tr>
<td>Exterior center</td>
<td>ExteriorCenter(G)</td>
</tr>
</tbody>
</table>

4. Conclusion

In this paper we have constructed appropriate programme using Groups, Algorithms and Programming (GAP) software to compute different characteristics such as structure of the group, order of the group, structure of the center, the number of conjugacy classes, commutator subgroup, abelianization, Whitehead`s universal quadratic functor and other characteristics of all finite nonabelianmetacyclicp-groups, where \( p \) is an odd prime, of nilpotency class at least three. In addition, such programme enables us to perform computation on structures of some other groups such as the nonabelian tensor square and various homological functors including Schur multiplier and exterior square. Furthermore, by computing the order of the epicenter or the order of the exterior Center, the capability can be determined.

To illustrate Theorem 1 and 2, the nonabelian tensor square of certain order groups are also computed in this paper. We found that the nonabelian tensor square of all computed groups of type (2.1) and type (2.2) is a direct product of four cyclic groups.

Finally, GAP codes for computing certain characteristics and subgroups given.

5. Acknowledgement

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6. References

3. J. B. Fraleigh (1982), First Course In Abstract Algebra, 3rd edn. Addison-Wesley, USA.