Abstract
In this paper we give new results on the best coapproximation in the Hausdorff topological vector space $X$. Assume that $f$ be a real valued function on $X$ and we present some results regarding $f$-best coapproximation. We determine under what conditions $f$-coproximinality can be transmitted to the quotient spaces and conversely.

2010 Mathematics Subject Classification: 41A52, 57N17.

Keywords: $f$-best Coapproximation, $f$-coChebyshev, $f$-compact, $f$-quasi-coChebychev

1. Introduction
The notion of $f$-best approximation in a vector space $X$ was given by Breckner and Brosowski. A few years later, T. D. Narang introduced the notions of $f$-best approximation and $f$-best coapproximation in a Hausdorff topological space. Recently the author in obtained some properties of $f$-best approximation sets in quotient spaces. In this paper we develop the theory of $f$-best coapproximation in quotient topological vector spaces. We want to determine under what conditions $f$-coproximinality can be transmitted to the quotient spaces and conversely.

Let $X$ be a Hausdorff topological vector space over $\mathbb{R}$ and $f$ a real valued function. Let $K$ be a nonempty closed subset of $X$ and $x \in X$. Element $k_0 \in K$ is said to be an $f$-best approximation to $x$ in $K$ if

$$f(x - k_0) = \inf\{f(x - k) : k \in K\}.$$

Furthermore, element $k_0 \in K$ is said to be an $f$-best coapproximation to $x$ in $K$ if

$$f(k - k_0) \leq f(x - k)$$

for all $k \in K$. We denote by $K_f^f(x)$ the collection of all such $k_0 \in K$. The set $K$ is said to be $f$-coproximinal if $K_f^f(x)$ is nonempty for each $x \in X$, and $f$-coChebysev if $K_f^f(x)$ is exactly singleton for each $x \in X$.

Putting

$$K_f^f = \{x \in X : f(k) \leq f(x - k) \text{ for all } k \in K\} = \left( K_f^f \right)^{-1}([0]).$$

It is clear that $k_0 \in K_f^f(x)$ if and only if $x - k_0 \in K_f^f$.

A subset $K$ of $X$ is called $f$-compact if every sequence $\{k_n\}$ in $K$ has a subsequence $\{k_{n_i}\}$ of $\{k_n\}$ and $k_0 \in K$ such that $f(k_{n_i} - k_0) \to 0$. $K$ is called $f$-quasi-coChebysev if $K_f^f(x)$ is nonempty and $f$-compact set in $X$ for every $x \in X$.

A function $f : X \to \mathbb{R}$ is absolutely homogeneous if $f(\alpha x) = |\alpha| f(x)$ for all $\alpha \in \mathbb{R}$ and $x \in X$.

When $X$ is a normed linear space over the filed of real numbers and $f(x, y) = \|x - y\|$ for all $x, y \in X$, the notions introduced above coincide with the corresponding notions that already exists in literature.

2. Set of $f$-Coapproximations
In this section, we give some characterizations of $f$-coproximinal sets in $X$.

Theorem 1
Let $X$ be a topological vector space and $f$ be a real valued function.
If $K$ is a subset of $X$, then

1. $R_{K+x+y}^f(x+y) = R_{K}^f(x) + y$ for all $x, y \in X$.
2. $K$ is $f$-coproximinal ($f$-coChebyshev) if and only if $K + y$ is $f$-coproximinal ($f$-coChebyshev) for every $y \in X$. Furthermore if $f$ is absolutely homogeneous, then
3. $R_{K}^f(\alpha x) = |\alpha| R_{K}^f(x)$ for all $x \in X$ and $\alpha \in \mathbb{R}$.
4. $K$ is $f$-coproximinal ($f$-coChebyshev) if and only if $\alpha K$ is $f$-coproximinal ($f$-coChebyshev) for all $\alpha \in \mathbb{R}$.

**Proof:**

(1) $k_0 + y \in R_{K+x+y}^f(x+y)$ if and only if $f(k_0 + y - (k + y)) \leq f(x + y - (k + y))$ for all $k \in K$, if and only if $f(k_0 - k) \leq f(x - k)$ for all $k \in K$, if and only if $k_0 \in R_{K}^f(x)$. Thus

$$R_{K+x+y}^f(x+y) = R_{K}^f(x) + y.$$

(2) It is clear by (2).

(3) If $\alpha = 0$, the result is true. Thus assume that $\alpha \neq 0$.

$k_0 \in R_{\alpha K}^f(\alpha x)$ if and only if $k_0 \in \alpha K$ and $f(k_0 - \alpha k) \leq f(\alpha x - \alpha k)$ for all $k \in K$, and if and only if $\frac{1}{\alpha} k_0 \in K$ and

$$|\alpha| f \left( \frac{1}{\alpha} k_0 - k \right) \leq |\alpha| f(x - k)$$

for all $k \in K$, and this implies that $\frac{1}{\alpha} k_0 \in R_{K}^f(x)$. So $k_0 \in \alpha R_{K}^f(x)$.

(4) It is clear by (3).

**Corollary 1**

Let $X$ be a topological vector space and $f$ be a real valued function. If $M$ be a nonempty subspace of $X$, then

(i) $R_{M}^f(x+y) = R_{M}^f(x) + y$, for every $x, y \in X$.

(ii) $R_{M}^f(\alpha x) = \alpha R_{M}^f(x)$ for every $x \in X$ and $\alpha \in \mathbb{R}[0]$.

**Proof:**

The proof is an immediate consequence of theorem (2.1) and this fact that $M + y = M$ and $\alpha M = M$ for all $y \in M$ and $\alpha \in \mathbb{R}[0]$.

**Theorem 2**

If $K$ is a subspace of $X$ and $f$ is a real function, then

(i) $K$ is $f$-coproximinal if and only if $X = K + K$. $f$.

(ii) $K$ is a $f$-coChebyshev subspace if and only if $X = K \oplus K$. $f$.

**Proof:**

(i) ($\Rightarrow$) Assume that $K$ is $f$-coproximinal, $x \in X$ and $k_0 \in R_{K}^f(x)$. Then, $x - k_0 \in K$. Hence $X = K + K$. $f$.

(ii) Let $x = K + K$. $f$. Then $x = k_0 + k$, $k_0 \in K, k \in K$. $f$ and so $0 \in R_{K}^f(k) = R_{K}^f(x - k_0) = R_{K}^f(x) - k_0$; hence $k_0 \in R_{K}^f(x)$. Therefore $K$ is $f$-coproximinal.

**Example 1**

Let $X = \mathbb{R}$ and $K \subseteq \{ (x, y) : x = y \}$, consider $f(x, y) = x^2 + y^2$, then $K = \{(x, y) : x = y \}$, and $R_{K}^f(a, a) = \left( \frac{a+b}{2}, \frac{a+b}{2} \right)$.

Therefore $K$ is $f$-coChebyshev.

**3. $f$-Coapproximation in Quotient Space**

Let $X$ be a topological vector space and $M$ be a closed subspace of $X$, and $f : X \rightarrow \mathbb{R}$ be a symmetric function (i.e., $f(-x) = f(x)$). Define

$$f(x + M) = \inf \{ f(x + y) \mid y \in M \}.$$

**Theorem 3**

Let $M$ be a closed subspace of $X$, and $K \supseteq M$ an $f$-coproximinal subspace of $X$. If $k_0 \in R_{K}^f(x)$, then $k_0 + M \in R_{K}^f(x + M)$. $M$
**Proof:**

Assume that $k_0 \in R^f_K (x)$ and $k_0 + M$ not in $R^f_M (x + M)$.

Then there exists $k' \in K$ such that

$$ \tilde{f}((x + M) - (k' + M)) < \tilde{f}((k' + M) - (k_0 + M)).$$

That is

$$ \tilde{f}((x - k') + M) < \tilde{f}((k' - k_0) + M).$$

Hence there exists $m \in M$ such that

$$ f((x - k') - m, t) < f((k' - k_0) + m).$$

Thus,

$$ f((x - (k' + m)) < f((k' + m) - k_0).$$

Therefore $k_0$ is not a $f$-best coapproximation to $x$ from $K$; which is a contradiction. So we have $k_0 + M \in R^f_M (x + M)$ and the proof is completed.

**Corollary 2**

Let $M$ be a closed subspace of $X$ and $K$ an $f$-coproximinal subspace of $X$ containing $M$. Then $K/M$ is an $f$-coproximinal subspace of $X/M$.

**Corollary 3**

Let $M$ be a closed subspace of $X$ and $K$ an $f$-coproximinal subspace of $X$ containing $M$. Then

$$ \pi\left(R^f_M (x)\right) \subseteq R^f_M (x + M).$$

**Theorem 4**

Let $M$ be a $f$-proximinal closed subspace of $X$ and $K$ a subspace of $X$ containing $M$. If $k_0 + M \in R^f_K (x + M)$, then there exists $m_0 \in M$ such that $k_0 + m_0 \in R^f_K (x)$.

**Proof:**

Let $k_0 + M \in R^f_K (x + M)$ where $k_0 \in K$. Then for every $k \in K$,

$$ \tilde{f}((x - M) - (k_0 - M)) \leq \tilde{f}((x - M) - (k - M)).$$

or

$$ \tilde{f}((x - k_0) + M) \leq \tilde{f}((x - k) + M).$$

By $f$-proximinality of $K$ in $M$, there exists $m_0 \in M$ such that

$$ f((k - k_0) - m_0) = \tilde{f}((k - k_0) + M).$$

Now we have

$$ f((k - (k_0 + m_0)) = f((k - k_0) - m_0)$$

$$ = \tilde{f}((k - k_0) + M))$$

$$ \leq \tilde{f}((x - k) + M)$$

$$ \leq f((x - k)$$

for every $k \in K$. Therefore $k_0 + m_0 \in R^f_K (x)$.

**Corollary 4**

Let $M$ be a $f$-proximinal closed subspace of $X$ and $K \supseteq M$ a subspace of $X$. If $K/M$ is $\tilde{f}$-coproximinal in $X/M$, then $K$ is $t$-coproximinal in $X$.

**Corollary 5**

Let $M$ be a $f$-proximinal closed subspace of $X$ and $K \supseteq M$ a subspace of $X$, then

$$ \pi\left(R^f_K (x)\right) = R^f_M (x + M).$$

**Proof:**

By corollary (3.3) we obtain

$$ \pi\left(R^f_K (x)\right) \subseteq R^f_M (x + M).$$

Also by theorem (3.1), $K/M$ is $\tilde{f}$-coproximinal in $X/M$.

Now let $k_0 + M \in R^f_K (x + M)$, where $k_0 \in K$. Now by theorem (3.4) $K$ is $f$-coproximinal and there exists $m_0 \in M$
such that $k_0 + m_n \in R^f_k(x)$, and so $k_0 + M \in \pi\left(R^f_k(x)\right)$; hence $R^f_k(x + M) \subseteq \pi\left(R^f_k(x)\right)$, and the proof is complete.

**Theorem 5**

Let $M$ be a $f$-proximinal closed subspace of $X$ and $K \supseteq M$ a subspace of $X$. If $K$ is $f$-coChebyshev then $\frac{K}{M}$ is a $\tilde{f}$-coChebyshev subspace of $\frac{X}{M}$.

**Proof:**

By theorem (3.1), $\frac{K}{M}$ is coproximinal. Let $x + M \in (X/M)(K/M)$ be arbitrary and $k_1 + M, k_2 + M \in R^f_k(x + M)$.

By theorem (3.4) there exists $m_1, m_2 \in M$ such that $k_1 + m_1, k_2 + m_2 \in R^f_k(x)$. Since $K$ is $f$-coChebyshev $k_1 + m_1 = k_2 + m_2$ and then $k_1 + M = k_2 + M$.

**Lemma 1**

Let $M$ be a $f$-proximinal closed subspace of $X$ and $K \supseteq M$ a subspace of $X$. If $\tilde{K}^f$ is convex, then $\left(\frac{K}{M}\right)^f = \left(\frac{R^f_k}{M}\right)^{-1}(M)$ is convex.

**Proof:**

Let $x + M, y + M \in \left(\frac{\tilde{K}^f}{M}\right)^f$ and $0 < \lambda < 1$. Then $M \in R^f_k(x + M)$ and $M \in R^f_k(y + M)$. Now from corollary (3.6) there exists $k_1 \in R^f_k(x)$ and $k_2 \in R^f_k(x)$ such that $\pi(k_1) = \pi(k_2) = M$. Therefore $x - k_1, y - k_2 \in \tilde{K}^f$. Since $\tilde{K}^f$ is convex then $\lambda(x - k_1) + (1 - \lambda)(y - k_2) \in \tilde{K}^f$. It follows that $\lambda k_1 + (1 - \lambda)k_2 \in \tilde{K}^f_k(\lambda x + (1 - \lambda) y)$; also

$\pi(\lambda k_1 + (1 - \lambda)k_2) = \lambda \pi(k_1) + (1 - \lambda)\pi(k_2) = \lambda M + (1 - \lambda) M = M$.

Now by corollary (3.6), $M \in R^f_k(\lambda x + (1 - \lambda) y + M)$; that is $\lambda(x - k_1) + (1 - \lambda)(y - k_2) \in \left(\frac{\tilde{K}}{M}\right)^f$. Therefore $\left(\frac{\tilde{K}}{M}\right)^f$ is convex.

**Theorem 6**

Let $M$ be a $f$-proximinal closed subspace of $X$ and $K \supseteq M$ be a $f$-coproximinal subspace of $X$ such that $K$ is $f$-quasi-coChebyshev. Then $\frac{K}{M}$ is $\tilde{f}$-quasi-coChebyshev.

**Proof:**

Since $K$ is $f$-coproximinal, therefore $\frac{K}{M}$ is $f$-coproximinal. Let $x + M \in \frac{X}{M}$ and $\{k_n + M\}$ be an arbitrary sequence in $R^f_k(x + M)$.

Then for every $k + M \in \frac{K}{M}$

$$\tilde{f}((k_n - k) + M) = \tilde{f}((k_n + M) - (k + M)) \leq \tilde{f}((x + M) - (k + M)) = \tilde{f}(x - k) + M \leq \tilde{f}(x - k).$$

Since $M$ is $f$-proximinal, by theorem (3.4) there exist $m_n \in M$ such that $k_n + m_n \in R^f_k(x)$, for all $n \geq 1$. Now since $R^f_k(x)$ is $f$-compact, there exists a subsequence $\{k_n + m_n\}$ such that it is $f$-converges to an element $\alpha_0 \in R^f_k(x)$; therefore $\alpha_0 + M \in R^f_k(x + M)$. It follows that $\{k_n + M\}$ is $f$-converges to an element $\alpha_0 + M$. Hence $\frac{K}{M}$ is quasi $\tilde{f}$-coChebyshev.

### 4. References