Abstract

The purpose of this paper is to prove some coupled fixed point theorems for mappings having the mixed monotone property in quasi-ordered metric spaces. Here, we take two mappings $F$ and $T$ in which $F$ is assumed to satisfy the mixed monotone property and $T$ is an ICS mapping. Also, an explained example is given to verify our findings.

Keywords: Contraction Type Mapping, Coupled Fixed Point, ICS Mapping, Mixed Monotone Property, Quasi-Ordered Metric Space

1. Introduction

The study of common fixed points of mappings satisfying certain contractive conditions has been at the core of dynamic research movements. The Banach fixed point theorem\(^1\) asserts that every contraction mapping in a complete metric space admits a unique fixed point. This theorem is an important tool in the theory of metric spaces and it has many applications in different fields of mathematics.

Fixed point theory in partially ordered metric spaces has greatly developed in recent time. Existence of a fixed point for contraction type mappings in partially ordered metric spaces and applications have been considered by many authors; for details\(^3^\)\(^-^4\).

Bhaskar and Lakshmikantham\(^5\) have introduced notions of a mixed monotone mapping and proved some coupled fixed point theorems. Afterwards some coupled fixed point theorems in partially ordered metric spaces were established by Lakshmikantham and Ciric\(^6\), B. Samet\(^7\) etc. Recently, Vishal Gupta et al.\(^8\)\(^-^9\) proved some common fixed point theorems in different spaces. Luong and Thuan\(^10\) presented some coupled fixed point theorems for a mixed monotone mapping in a partially ordered metric space which are generalization of the results of Bhaskar and Lakshmikantham\(^5\).

In this paper, we are continuously going to study the existence problems of coupled fixed points for mixed monotone mapping and an injective, continuous mapping on a quasi-ordered metric space.

2. Preliminaries

2.1 Definition 1

Let $X$ be a nonempty set and $\leq$ be a quasi-order (that is, a reflexive and transitive relation) on $X$ then $(X, \leq)$ is called a quasi-order set.

2.2 Definition 2

Let $X$ be a nonempty set. A real-valued function $d : X \times X \to \mathbb{R}$ is said to be quasi-metric on $X$ if

\begin{align*}
(1) \quad & d(x, y) \geq 0 \quad \text{for all } x, y \in X \\
(2) \quad & d(x, y) = 0 \quad \text{if } x = y \\
(3) \quad & d(x, y) \leq d(x, z) + d(z, y).
\end{align*}

Then the pair $(X, d)$ is called a quasi-metric space.
2.3 Definition 3
Let \((X, \leq)\) be a quasi-ordered set and \(F : X \times X \to X\) be a map. The mapping \(F\) is said to have mixed monotone property on \(X\) if \(F\) is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any \(x, y \in X\),
\[x_1, x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y),\]
and
\[y_1, y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).\]

2.4 Definition 4
Let \(X\) be a non-empty set and \(F : X \times X \to X\) be a map. We call an element \((x, y) \in X \times X\) a coupled fixed point of \(F\) if \(F(x, y) = x\) and \(F(y, x) = y\).

2.5 Definition 5
Let \((X, d)\) be a metric space with quasi-order \(\leq\). A mapping \(T : X \to X\) is said to be ICS if \(T\) is injective, continuous and the property that, for every sequence \(\{x_n\}\) in \(X\), if \(\{T x_n\}\) is convergent then \(\{x_n\}\) is also convergent.

Consider \((X, d)\) be a metric space with a quasi-ordered \(\leq\), we endow the product space \(X \times X\) with the metric \(\rho\) defined by \(\rho((x, y), (u, v)) = \frac{d(x, u) + d(y, v)}{2}\) for any \((x, y), (u, v) \in X \times X\).

And let \(\Omega\) denote the class of all those functions \(\delta : [0, \infty) \to [0, 1)\), such that \(\delta(\omega_n) \to 1\) implies \(\omega_n \to 0\).

Theorem 1
Let \((X, d, \leq)\) be a sequentially complete metric space and \(T : X \times X \to X\) be a continuous map having the mixed monotone property on \(X\). Assume that there exists \(\delta \in \Omega\) such that for any \((x, y)\), \((u, v) \in X \times X\) with \((u, v) \leq (x, y)\),
\[d(T(x, y), T(u, v)) \leq \delta(\rho((x, y), (u, v))) \rho((x, y), (u, v)).\]

If there exist \(x_0, y_0 \in X\) such that \(x_0 \leq T(x_0, y_0)\) and \(y_0 \leq T(y_0, x_0)\) then there exist \(x^*, y^* \in X\) such that \(x^* = T(x^*, y^*)\) and \(y^* = T(y^*, x^*)\).

In addition to the conditions in Theorem 1, we have taken one ICS map to prove a coupled fixed point result. Moreover, Example 1 justifies the results.

Main Results
Theorem 2
Let \((X, d, \leq)\) be a sequentially complete metric space and \(F : X \times X \to X\) be a continuous map having the mixed monotone property on \(X\) and \(T : X \to X\) is an ICS mapping such that for any \((x, y), (u, v) \in X \times X\) with \((u, v) \leq (x, y)\),
\[d(TF(x, y), TF(u, v)) \leq \delta(\rho(Tx, Ty), (Tu, Tv))) \rho((Tx, Ty), (Tu, Tv)),\]
(3.1)
where \(\rho\) be the metric defined on product space \(X \times X\) and \(\delta \in \Omega\). If there exists \(x_0, y_0 \in X\) such that \(x_0 \leq F(x_0, y_0)\) and \(y_0 \geq F(y_0, x_0)\), then there exist \(x, y \in X\) such that \(x = F(x, y)\) and \(y = F(y, x)\).

Proof: Let \(x_0, y_0 \in X\) be such that \(x_0 \leq F(x_0, y_0)\) and \(y_0 \geq F(y_0, x_0)\). We can choose \(x_1, y_1 \in X\) such that \(x_1 = F(x_0, y_0)\) and \(y_1 = F(y_0, x_0)\). Again we can choose \(x_2, y_2 \in X\) such that \(x_2 = F(x_1, y_1)\) and \(y_2 = F(y_1, x_1)\). Continuing this process we can construct sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that
\[x_{n+1} = F(x_n, y_n)\text{ and } y_{n+1} = F(y_n, x_n)\text{ for all } n \geq 0.\]
(3.2)

Further, for \(n = 1, 2, 3, \ldots\), we have
\[x_{n+1} = F^{n+1}(x_0, y_0)\text{ and } y_{n+1} = F^{n+1}(y_0, x_0)\]
and
\[x_n \leq x_{n+1} \text{ and } y_n \geq y_{n+1} \text{ for all } n \geq 0.\]
(3.3)

For \(n = 0\), we have \(x_0 \leq F(x_0, y_0)\) and \(y_0 \geq F(y_0, x_0)\) and also \(x_1 = F(x_0, y_0)\) and \(y_1 = F(y_0, x_0)\), we have \(x_0 \leq x_1\) and \(y_0 \geq y_1\).

Thus (3.3) hold for \(n = 0\). Now, suppose that (3.3) hold for some fixed \(n \geq 0\). Then since \(x_n \leq x_{n+1}\) and \(y_n \geq y_{n+1}\), and as \(F\) has the mixed monotone property, from (3.2), we have
This inequality yields \( \lim_{n \to \infty} \delta(d_n) = 1 \).

And since \( \delta \in \Omega \), this implies \( d = 0 \). Therefore,

\[
\lim_{n \to \infty} \rho\left(\left(\text{Tx}_{n}, \text{Ty}_{n}\right), \left(\text{Tx}_{n+1}, \text{Ty}_{n+1}\right)\right)
= \lim_{n \to \infty} \frac{1}{2} \left[ d\left(\text{Tx}_{n}, \text{Tx}_{n+1}\right) + d\left(\text{Ty}_{n}, \text{Ty}_{n+1}\right) \right] = 0
\]

So, \( \lim_{n \to \infty} \left[ d\left(\text{Tx}_{n}, \text{Tx}_{n+1}\right) + d\left(\text{Ty}_{n}, \text{Ty}_{n+1}\right) \right] = 0 \) (3.7)

Now, we prove that \( \{\text{Tx}_{n}\} \) and \( \{\text{Ty}_{n}\} \) are Cauchy sequences, suppose to contrary, that at least one of \( \{\text{Tx}_{n}\} \) or \( \{\text{Ty}_{n}\} \) is not a Cauchy sequence. Then there exist \( \varepsilon > 0 \), for which we can find subsequences \( \{\text{Tx}_{n(k)}\} \), \( \{\text{Ty}_{n(k)}\} \) of \( \{\text{Tx}_{n}\} \) and \( \{\text{Ty}_{n(k)}\} \), \( \{\text{Ty}_{m(k)}\} \) of \( \{\text{Ty}_{n}\} \) with \( m(k) > n(k) \geq k \) such that

\[
d\left(\text{Tx}_{n(k)}, \text{Tx}_{m(k)}\right) + d\left(\text{Ty}_{n(k)}, \text{Ty}_{m(k)}\right) \geq \varepsilon \quad \text{for} \quad k = 1, 2, 3, ...
\]

and

\[
d\left(\text{Tx}_{n(k)}, \text{Tx}_{m(k)-1}\right) + d\left(\text{Ty}_{n(k)}, \text{Ty}_{m(k)-1}\right) < \varepsilon
\]

From (3.8), (3.9) and by triangle inequality,

\[
\varepsilon \leq d\left(\text{Tx}_{n(k)}, \text{Tx}_{m(k)-1}\right) + d\left(\text{Tx}_{m(k)-1}, \text{Tx}_{m(k)}\right) + d\left(\text{Ty}_{n(k)}, \text{Ty}_{m(k)-1}\right) + d\left(\text{Ty}_{m(k)-1}, \text{Ty}_{m(k)}\right)
= d\left(\text{Tx}_{n(k)}, \text{Tx}_{m(k)}\right) + 2d\left(\text{Ty}_{n(k)}, \text{Ty}_{m(k)}\right) + 2d\left(\text{Ty}_{m(k)}, \text{Ty}_{m(k)-1}\right) + 2d\left(\text{Ty}_{m(k)-1}, \text{Ty}_{m(k)}\right)
< \varepsilon + 2d\left(\text{Ty}_{m(k)}, \text{Ty}_{m(k)-1}\right) + 2d\left(\text{Ty}_{m(k)-1}, \text{Ty}_{m(k)}\right)
\]

Taking the limit as \( k \to \infty \) and using (3.7)

\[
\lim_{k \to \infty} \left[ d\left(\text{Tx}_{n(k)}, \text{Tx}_{m(k)}\right) + d\left(\text{Ty}_{n(k)}, \text{Ty}_{m(k)}\right) \right] = \varepsilon
\]

Again the triangle inequality gives

\[
\varepsilon \leq d\left(\text{Tx}_{n(k)}, \text{Tx}_{n(k)-1}\right) + d\left(\text{Tx}_{n(k)-1}, \text{Tx}_{m(k)}\right) + d\left(\text{Ty}_{n(k)}, \text{Ty}_{m(k)-1}\right) + d\left(\text{Ty}_{m(k)-1}, \text{Ty}_{m(k)}\right) + d\left(\text{Ty}_{m(k)}, \text{Ty}_{m(k)-1}\right) + d\left(\text{Ty}_{m(k)-1}, \text{Ty}_{m(k)}\right);
\]
Vol 8 (1), then there exist 
(3.13)
\[x \in \Omega, \text{ such that for any } x \in \Omega, \text{ implies }\]
\[\lim_{n \to \infty} x_n = x \text{ and } \lim_{n \to \infty} y_n = y. \tag{3.12}\]

Again since \(F\) is assumed to be continuous mapping, we have
\[x = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} F(x_n, y_n) = F(x, y),\]
and
\[y = \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} F(y_n, x_n) = F(y, x).\]

This proves that \((x, y)\) is a coupled fixed point of \(F\).

**Theorem 3**

Let \((X, d, \leq)\) be a sequentially complete metric space and \(F : X \times X \to X\) be a map having the mixed monotone property on \(X\). Let \(T : X \to X\) be an ICS mapping. Assume that

- Any non-decreasing sequence \(\{x_n\}\) with \(x_n \to x\) implies \(x_n \leq x\) for each \(n \in \mathbb{N}\).
- Any non-increasing sequence \(\{y_n\}\) with \(y_n \to y\) implies \(y_n \leq y\) for each \(n \in \mathbb{N}\).

Further let there exist \(\delta \in \Omega\) such that for any \((x, y), (u, v) \in X \times X\) with \((u, v) \leq (x, y),\)
\[d(F(x, y), F(u, v)) \leq \delta(\rho((x, y), (u, v))) \rho((x, y), (u, v)).\]

If there exist \(x_0, y_0 \in X\) with \(x_0 \leq F(x_0, y_0)\) and \(y_0 \leq F(y_0, x_0)\), then there exist \(x, y \in X\) such that \(x = F(x, y)\) and \(y = F(y, x)\).

**Proof:** According to Theorem 2, there exist a non-decreasing sequence \(\{x_n\}\) and non-increasing sequence \(\{y_n\}\) such that \(x_n \to x\) and \(y_n \to y\), from (i) and (ii) we have \(x_n \leq x\) and \(y_n \leq y\) for all \(n \in \mathbb{N}\). Then by triangle inequality and (3.13) we get
\[d(Tx, TF(x, y)) \leq d(Tx, Tx_{n+1}) + d(Tx_{n+1}, TF(x, y))\]
\[= d(Tx, Tx_{n+1}) + d(TF(x_n, y_n), TF(x, y))\]
\[\leq d(Tx, Tx_{n+1}) + \delta \left( \frac{1}{2} (d(x_n, x) + d(y_n, y)) \right)\]
\[\leq d(Tx, Tx_{n+1}) + \frac{1}{2} (d(x_n, x) + d(y_n, y))\]

On taking \(n \to \infty\), we obtain \(d(Tx, TF(x, y)) \leq 0\) and hence \(Tx = TF(x, y)\).
Similarly, we have $T y = TF (y, x)$. Since $T$ is injective mapping, we have $x = F (x, y)$ and $y = F (y, x)$.

**Theorem 4**

In addition to the hypothesis of Theorem 2 suppose that for every $(x, y), (x_1, y_1) \in X \times X$, there exist an $(u, v) \in X \times X$ that is comparable to $(x, y)$ and $(x_1, y_1)$, then $F$ has a unique coupled fixed point.

Proof: From Theorem 2 the set of coupled fixed point is non-empty. Suppose $(x, y)$ and $(x_1, y_1)$ are coupled fixed point of $F$, that is

$$x = F (x, y), y = F (y, x), x_1 = F (x_1, y_1) \text{ and } y_1 = F (y_1, x_1).$$

We claim that $x = x_1$ and $y = y_1$.

By assumption, there exist $(u, v) \in X \times X$ that is comparable to $(x, y)$ and $(x_1, y_1)$, we define the sequences $\{u_n\}$ and $\{v_n\}$ as follows

$$u_0 = u, v_0 = v, u_{n+1} = F (u_n, v_n) \text{ and } v_{n+1} = F (v_n, u_n)$$

for all $n \in N$.

Since $(u, v)$ is comparable with $(x, y)$, we may assume that $(x, y) \geq (u_n, v_n)$ for all $n \in N$. (3.14)

Suppose that (3.14) holds for some $n \geq 0$. Then by mixed monotone property of $F$ we have

$$u_{n+1} = F (u_n, v_n) \leq F (x, y) = x,$$

and

$$v_{n+1} = F (v_n, u_n) \leq F (y, x) = y.$$

That is, $(x, y) \geq (u_{n+1}, v_{n+1})$, therefore (3.18) holds.

From condition (3.1) of Theorem 2, we have

$$d (Tx, Tu_n) = d (TF (x, y), TF (u_{n-1}, v_{n-1}))$$

$$\leq \delta \left( \rho \left( \left(Ty, fy\left(Tu_{n-1}, Tv_{n-1}\right)\right)\right) \right)$$

$$\rho \left( \left(Tx, fy\left(Tu_{n-1}, Tv_{n-1}\right)\right)\right)$$

$$= \frac{1}{2} \left( d (Tx, Tu_{n-1}) + d (Ty, Tv_{n-1}) \right)$$

and

$$d (Tv_n, Ty) = d (\left(Tu_{n-1}, u_{n-1}\right), TF (y, x))$$

$$\leq \delta \left( \rho \left( \left(Tv_{n-1}, u_{n-1}\right), (Ty, Tx)\right)\right)$$

$$\rho \left( \left(Tv_{n-1}, u_{n-1}\right), (Tv_{n-1}, u_{n-1})\right), (Ty, Tx)\right)$$

$$= \frac{1}{2} \left( d (Tv_{n-1}, Ty) + d (Tv_{n-1}, Tx) \right)$$

Adding the above inequalities, we get

$$d (Tx, Tu_n) + d (Ty, Tv_n) \leq d (Tx, Tu_{n-1})$$

$$+ d (Ty, Tv_{n-1})$$

(3.15)

Set $\sigma_n = d (Tx, Tu_n) + d (Ty, Tv_n)$, it follows from (3.15) that $\{d_n\}$ is monotone decreasing sequence of positive real numbers, therefore there is some $\sigma \geq 0$ such that

$$\lim_{n \to \infty} \sigma_n = \lim_{n \to \infty} \left[ d (Tx, Tu_n) + d (Ty, Tv_n) \right] = \sigma$$

Suppose that $\sigma > 0$, taking $n \to \infty$ in (3.15) we have

$$\sigma = \lim_{n \to \infty} \left[ d (Tx, Tu_n) + d (Ty, Tv_n) \right]$$

$$\leq \lim_{n \to \infty} \left[ d (Tx, Tu_{n-1}) + d (Ty, Tv_{n-1}) \right]$$

$$= \lim_{n \to \infty} \sigma_{n-1}$$

$$= \sigma$$

This is a contradiction.

Thus, $\sigma = 0$.

Therefore, $\lim_{n \to \infty} \left[ d (Tx, Tu_n) + d (Ty, Tv_n) \right] = 0$

This implies $\lim_{n \to \infty} \left[ d (Tx, Tu_n) \right] = \lim_{n \to \infty} \left[ d (Ty, Tv_n) \right] = 0$ (3.16)

Similarly, we have $\lim_{n \to \infty} \left[ d (Tx, Tu_n) \right] = \lim_{n \to \infty} \left[ d (Ty, Tv_n) \right] = 0$ (3.17)

From (3.16) and (3.17), we have

$$Tx = Tx_1 \text{ and } Ty = Ty_1.$$
Since, \( T \) is injective mapping therefore \( x = x_1 \) and \( y = y_1 \). Hence the result.

**Example 1.** Let \( X = \{0,1,2,3,4\ldots\} \) and \( d(x,y) = |x - y| \) be the usual metric.

Define a quasi order as follows:
\[ x \leq y \text{ if and only if } x = y \text{ or } \{x,y\} \in \{0,1,2,3\} \]

Then \((X,d)\) is a complete metric space and \((X,\leq)\) is a quasi-ordered set i.e. for all \( a,b,c \in X \)
\[ a \leq a \text{ for all } a \in X \text{ (reflexive) and } a \leq b \text{, } b \leq c \text{ implies } a \leq c \text{ (transitive)}. \]

Consider \( A = \{(2,0),(2,1)\}, B = \{(3,0),(3,1),(3,2)\} \)

and \( C = \{(4,0),(4,1),(4,2),(4,3)\} \)

Also \( F: X \times X \to X \) be defined by
\[ F(x,y) = \begin{cases} 
1 & \text{for } (x,y) \in A \\
2 & \text{for } (x,y) \in B \\
3 & \text{for } (x,y) \in C \\
0 & \text{for } (x,y) \in X^2 - (A \cup B \cup C) 
\end{cases} \]

Then \( F \) has mixed monotone property and there exist \( x_0 = 0 \) and \( y_0 = 3 \) such that
\[ x_0 \leq F(x_0,y_0) \text{ and } y_0 \geq F(y_0,x_0). \]

Define \( \delta: [0,\infty) \to [0,1) \) by \( \delta(t) = \frac{11}{25} \)

Now, the condition
\[ d(F(x,y),F(u,v)) \leq \delta \left( \frac{d(x,u) + d(y,v)}{2} \right) \left( \frac{d(x,u) + d(y,v)}{2} \right) \]  \hspace{1cm} (3.18)

is not true.

Indeed, for \( x = 2, y = 0, u = 1, v = 3 \)
\[ F(x,y) = F(2,0) = 1 \text{ and } F(u,v) = F(1,3) = 0 \]

L.H.S of (3.18) becomes \( d (1,0) = 1 \)

and R.H.S is \( \delta \left( \frac{d(2,1) + d(0,3)}{2} \right) \left( \frac{d(2,1) + d(0,3)}{2} \right) = 2\delta(2) \)
\[ = 2 \times \frac{11}{25} = \frac{22}{25} \]

Now, we can show that the condition of our theorem is true by taking one more map
\[ T: X \times X \to X \text{ by } T0 = 0, T1 = 1, T2 = 5, T3 = 6, T4 = 9, T5 = 1 \]

Now we show \( F \) and \( T \) satisfy:
\[ d(TF(x,y),TF(u,v)) \leq \delta \left( \frac{d(Tx,Tu) + d(Ty,Tv)}{2} \right) \]
\[ \left( \frac{d(Tx,Tu) + d(Ty,Tv)}{2} \right) \]  \hspace{1cm} (3.19)

We discuss the result in the following cases:

**Case 1:** When \((x, y) = (2, 0)\), then \((u, v) = \{(0,0),(0,1), \ldots,(2,0),(2,1),(2,2)\}\)
\[ \text{Taking } (x, y) = (2, 0) \text{ and } F(u, v) = (0, 0), \text{ we get} \]

L.H.S of (3.19) is \( d(TF(2,0),TF(0,0)) = d(T1,T0) = d(1,0) = 1 \)

and R.H.S is \( \delta \left( \frac{d(T2,T0) + d(T0,T0)}{2} \right) \left( \frac{d(T2,T0) + d(T0,T0)}{2} \right) = \delta \left( \frac{5}{2} \right) \left( \frac{5}{2} \right) = \frac{25}{2} \times \frac{5}{2} = \frac{40}{1} > 1 \]

Similarly, for other values of \((u, v)\) the condition (3.19) is satisfied.

**Case 2:** When \((x, y) = (2, 1)\), then \((u, v) = \{(0,1),(0,2), \ldots,(2,0),(2,1),(2,2)\}\)
\[ \text{On taking } (x, y) = (2, 1) \text{ and } (u, v) = (1, 3) \]

L.H.S of (3.25) is \( d(TF(2,1),TF(1,3)) = d(T1,T0) = 1 \)

R.H.S is \( \delta \left( \frac{d(T2,T1) + d(T1,T3)}{2} \right) \left( \frac{d(T2,T1) + d(T1,T3)}{2} \right) = 6\delta(6) = \frac{66}{25} > 1 \]

Thus, in all cases the condition (3.19) is satisfied.

### 3. References