Abstract

Background/Objectives: To formulate and analyze a vector host epidemic model with non-monotonic and bilinear incidences. Methods/Statistical Analysis: The stability conditions of disease free equilibrium and endemic equilibrium are investigated by constructing suitable Lyapunov functions. Numerical simulation is carried out to justify the theoretical results. Results/Findings: The disease becomes endemic when the basic reproduction number is greater than one and it fades out when it is less than one. Conclusion/Application: In endemic state of the disease, number of infective host decreases as awareness of vaccination and preventive measures increases and number of vectors approaches zero as the awareness of use of insecticides and cleanliness tends to infinity.

Keywords: Basic Reproduction Number, Disease Free Equilibrium, Endemic Equilibrium, Stability, Vector Borne Diseases

1. Introduction

Vectors become the main mediators of infectious diseases to the host population. Awareness about preventive measures to reduce vector-host contacts and the insecticide control of the vector helps to eradicate the disease. Infectious diseases have major social and economical impact in the population. Mathematical models on epidemiology provide significant insight into population behavior and control. They play an important role in understanding the dynamics of the diseases and to make the public health policies for controlling disease.

Our aim is to develop a vector born disease model with non-monotonic incidence rate and bilinear incidence rate which describes the effect of social awareness on epidemics. In section 2, we formulate a system of differential equations which represent the vector host epidemiological model with non-monotonic rate and bilinear incidence rate and compute the basic reproduction number. We also obtain the disease free equilibrium and endemic equilibrium for the model and analyze the stability conditions for these equilibriums in section 3. In section 4, we show the outcomes using numerical simulation.
Stability Analysis of a Vector-borne Disease Model with Nonlinear and Bilinear Incidences

\[
\begin{align*}
\frac{dS(t)}{dt} &= \alpha_1 - \frac{k\lambda S(t)V(t)}{1 + aV(t) + bV^2(t)} - \beta_1 S(t) \\
\frac{dI(t)}{dt} &= \frac{k\lambda S(t)V(t)}{1 + aV(t) + bV^2(t)} - \gamma I(t) - \beta_1 I(t) \\
\frac{dR(t)}{dt} &= \gamma I(t) - \beta_2 R(t) \\
\frac{dM(t)}{dt} &= \alpha_2 - \frac{k\lambda S(t)V(t)}{1 + aV(t) + bV^2(t)} - \beta_1 M(t) \\
\frac{dV(t)}{dt} &= \lambda_3 M(t)I(t) - \beta_2 V(t)
\end{align*}
\]

Therefore we get
\[
\begin{align*}
\frac{d}{dt} N_1(t) &= \alpha_1 + \beta_2 N_1(t) \\
\frac{d}{dt} N_2(t) &= \alpha_2 + \beta_2 N_2(t)
\end{align*}
\]

The total population sizes of host and vector are asymptotically constant.

\[
\text{i.e. } \lim_{t \to \infty} N_1(t) = \frac{\alpha_1}{\beta_1}, \quad \lim_{t \to \infty} N_2(t) = \frac{\alpha_2}{\beta_2}
\]

Without loss of generality, we assume that \(N_1(t) = \frac{\alpha_1}{\beta_1}, \quad N_2(t) = \frac{\alpha_2}{\beta_2}\) for all \(t \geq 0\).

Therefore the dynamical system (1.1) is now equivalent to the dynamical system given by
\[
\begin{align*}
\frac{dS(t)}{dt} &= \alpha_1 - \frac{k\lambda S(t)V(t)}{1 + aV(t) + bV^2(t)} - \beta_1 S(t) \\
\frac{dI(t)}{dt} &= \frac{k\lambda S(t)V(t)}{1 + aV(t) + bV^2(t)} - \gamma I(t) - \beta_1 I(t) \\
\frac{dV(t)}{dt} &= \lambda_3 M(t)I(t) - \beta_2 V(t)
\end{align*}
\]

The values of \(R\) and \(M\) are determined from
\[
R = \frac{\alpha_1}{\beta_1} - S - I \quad \text{and} \quad M = \frac{\alpha_2}{\beta_2} - V,
\]

We need non-negative solutions for biological reasons. Due to mathematical properties of the solutions we study the system (2.2) in the closed set
\[
\Gamma = \left\{(S, I, V) \in \mathbb{R}_+^3 : 0 \leq S + I \leq \frac{\alpha_1}{\beta_1}, \quad 0 \leq V \leq \frac{\alpha_2}{\beta_2}, S \geq 0, I \geq 0 \right\}
\]

System (2.2) has disease free equilibrium \(E_0\left(\frac{\alpha_1}{\beta_1}, 0, 0\right)\).

To find the endemic equilibrium of system (2.2), set
\[
\begin{align*}
\alpha_1 - \frac{k\lambda S(t)V(t)}{1 + aV(t) + bV^2(t)} - \beta_1 S(t) &= 0 \\
\frac{k\lambda S(t)V(t)}{1 + aV(t) + bV^2(t)} - \gamma I(t) - \beta_1 I(t) &= 0 \\
\lambda_3 M(t)I(t) - \beta_2 V(t) &= 0
\end{align*}
\]

This gives
\[
b\beta_2 \gamma (\gamma + \beta_1) V + \beta_1 \gamma (\gamma + \beta_1) - k\lambda \alpha_2 \lambda \lambda = 0
\]

The basic reproduction number is defined as follows
\[
R_0 = \frac{k\alpha_1 \alpha_2 \lambda \lambda}{\beta_2 (\gamma + \beta_1)}
\]

The endemic equilibrium \(E'(S', I', V')\) is given by the following equations
\[
S' = \left(\frac{\gamma + \mu_1}{\beta_2 V^*} + \frac{\beta_2 V^*}{k\lambda_3 \alpha_2 - \beta_2 V^*}\right)
\]

\[
I' = \frac{\beta_2 V^*}{\lambda_3 \alpha_2 - \beta_2 V^*}
\]

\[
V^* = \frac{-\Delta + \sqrt{\Delta^2 - 4b\beta_2 \gamma (\gamma + \beta_1)^2 (1 - R_0)}}{2b\beta_2 \gamma (\gamma + \beta_1)}
\]

where \(\Delta = k\alpha_1 \beta_2 (\gamma + \mu_1) + \alpha_1 \beta_1 \beta_2 (\gamma + \beta_1)\)

3. Stability Analysis

In this section, we derive the stability conditions for the disease-free and the endemic equilibrium of model (2.2).

3.1 Theorem

If \(R_0 < 1\) then the disease free equilibrium \(E_0\) is locally asymptotically stable and it is unstable for \(R_0 > 1\).

Proof: For the disease free equilibrium \(E_0\), we obtain the Jacobian matrix as:
Characteristic equation of this matrix is given by
\[
(\xi + \beta_1) \left[ \xi^2 + (\gamma + \beta_1 + \beta_2) \xi + (\gamma + \beta_1) \beta_2 (1 - R_0) \right] = 0
\]

Clearly, this characteristic equation has a root \( \xi = -\beta_1 \) which is negative. Another two roots are determined by the equation
\[
\xi^2 + (\gamma + \beta_1 + \beta_2) \xi + (\gamma + \beta_1) \beta_2 (1 - R_0) = 0
\]

By Routh - Hurwitz criterion\(^5,6\), we have that if \( R_0 < 1 \) then above equation has both the roots with negative real part and if \( R_0 > 1 \) then it has one of the root with negative real part and another with positive real part. Thus, we obtain that all the Eigen values of the above characteristic equation have negative real part for \( R_0 < 1 \) and hence local asymptotically stability of the disease free equilibrium \( E_0 \) is established. When \( R_0 > 1 \), two of the Eigen values are negative and one is positive and so \( E_0 \) is unstable in this case. Hence the theorem follows.

### 3.2 Theorem

The disease free equilibrium \( E_0 \) is globally asymptotically stable for \( R_0 \leq 1 \) and if \( R_0 > 1 \) then it is unstable.

**Proof:** We consider the function \( L_1 \) as follows,
\[
L_1 = \frac{k_\lambda \alpha}{\beta_1 \beta_2} V + I
\]

\[
\frac{dL_1}{dt} = \frac{k_\lambda \alpha \gamma}{\beta_1 \beta_2} \left[ \lambda \left( \frac{\alpha}{\beta_2} - V \right) I - \mu_1 V \right] + \frac{k_\lambda \gamma S V}{1 + aV + bV^2} - (\gamma + \beta) I
\]

\[
= -(\gamma + \beta) \left[ 1 - \frac{k_\lambda \gamma \alpha \lambda}{\beta_1 \beta_2 (\gamma + \beta)} \right] I \frac{\alpha}{\beta_2} V - \frac{k_\lambda \gamma S V}{1 + aV + bV^2}
\]

\[
\leq -(\gamma + \beta) \left[ 1 - \frac{k_\lambda \gamma \alpha \lambda}{\beta_1 \beta_2 (\gamma + \beta)} \right] V I, \quad \text{as} \quad S \leq \frac{\alpha}{\beta_1}
\]

Since at all \( t \geq 0 \), if \( R_0 \leq 1 \) then \( \frac{dL_1}{dt} \leq 0 \), the function \( L_1 \) is a Lyapunov function. Also, at the disease free equilibrium i.e. at \( E_0 \) \((S/\beta_1, 0, 0)\) we find \( \frac{dL_1}{dt} = 0 \). Thus, \( E_0 \) is the largest invariant set in the closed set \( \Gamma \). Therefore, \( E_0 \) is globally stable using LaSalle's invariance principle\(^7\).

We now investigate the local stability of endemic equilibrium and derive the global stability of endemic equilibrium in the feasible region \( \Gamma \) by proving uniform persistence of the system (2.2). We also apply theory of second compound equations to show asymptotic orbital stability of periodic solutions.

### 3.3 Theorem

The endemic equilibrium \( E^* \) of the system (2.2) is locally asymptotically stable if \( R_0 > 1 \).

**Proof:** We have
\[
\frac{dL_1}{dt} = -\frac{k_\lambda \gamma \alpha \lambda}{\beta_1 \beta_2 (\gamma + \beta)} \left[ \lambda \left( \frac{\alpha}{\beta_2} - V \right) I - \mu_1 V \right] + \frac{k_\lambda \gamma S V}{1 + aV + bV^2}
\]

\[
= -\frac{k_\lambda \gamma \alpha \lambda}{\beta_1 \beta_2 (\gamma + \beta)} \left[ \lambda \left( \frac{\alpha}{\beta_2} - V \right) I - \mu_1 V \right] + \frac{k_\lambda \gamma S V}{1 + aV + bV^2}
\]

\[
\leq -(\gamma + \beta) \left[ 1 - \frac{k_\lambda \gamma \alpha \lambda}{\beta_1 \beta_2 (\gamma + \beta)} \right] V I, \quad \text{as} \quad S \leq \frac{\alpha}{\beta_1}
\]

Second additive compound matrix \( J(E^*) \) is given by
\[
\int \frac{dJ(E^*)}{dt} = \frac{k_\lambda \gamma \alpha \lambda}{\beta_1 \beta_2 (\gamma + \beta)} \left[ \lambda \left( \frac{\alpha}{\beta_2} - V \right) I - \mu_1 V \right] + \frac{k_\lambda \gamma S V}{1 + aV + bV^2}
\]

\[
\leq -(\gamma + \beta) \left[ 1 - \frac{k_\lambda \gamma \alpha \lambda}{\beta_1 \beta_2 (\gamma + \beta)} \right] V I, \quad \text{as} \quad S \leq \frac{\alpha}{\beta_1}
\]

We have, \( \text{tr}(J(E^*)), \det(J(E^*)) \) and \( \text{det}(J(E^*)) \) all negative. Hence, all Eigen values of \( J(E^*) \) have negative real part\(^8\). Hence the theorem.
3.4 Theorem

If \( R_0 > 1 \), then system (2.2) is uniformly persistent, that is, there exists \( \varepsilon > 0 \) (independent of initial conditions), such that \( \lim\inf_{t \to \infty} S(t) > \varepsilon, \lim\inf_{t \to \infty} I(t) > \varepsilon \) and \( \lim\inf_{t \to \infty} V(t) > \varepsilon \).

**Proof:** To prove this, we show the following results:

- For system (2.2), \( E_0 \) is only one omega-limit point on the boundary of \( \Gamma \).
- For \( R_0 > 1 \), \( E_0 \) cannot be the omega-limit point of any orbit in \( \text{Int} \Gamma \).

(i) The vector field is transversal to the boundary of \( \Gamma \), except in the \( S \)-axis, which is invariant for the system (2.2). On the \( S \)-axis, we have

\[
\frac{dS}{dt} = \alpha_1 - \beta_1 S
\]

It follows from the above expression that \( S \to \frac{\alpha_1}{\beta_1} \) as \( t \to \infty \). So, the first part is proved.

(ii) Now we define the following function in \( \Gamma \).

\[
L_1(t) = \frac{\beta_1(1 + R_0)}{2k\lambda_1\alpha_1} I(t) + V(t)
\]

\[
\frac{dL_1(t)}{dt} = \beta_1(1 + R_0) \left( \frac{k\lambda_1 S(t)V(t)}{1 + aV(t) + bV^2(t)} - \gamma I(t) - \beta_1 I(t) \right)
+ \lambda_1 \left( \frac{\alpha_1}{\beta_1} - V(t) \right) I(t) - \beta_1(1 + R_0) \left( \frac{2\alpha_1}{2R_0} - \frac{1}{1 + aV(t) + bV^2(t)} \right) V(t)
+ \beta_1(1 + R_0) \left[ S(t) - \frac{2\alpha_1}{\beta_1(1 + R_0)} \right] V(t) + \lambda_1 \left[ \frac{\alpha_1}{\beta_1} - V(t) - \frac{1}{2} \right] \frac{1}{R_0} I(t)
\geq \beta_1(1 + R_0) \left[ \frac{a_1}{\beta_1} - V(t) - \frac{1}{2} \right] \frac{1}{R_0} I(t)
\]

Since \( R_0 > 1 \), then \( \frac{1}{2} \frac{1}{R_0} + 1 < 1 \) and \( \frac{2}{1 + R_0} < 1 \). Hence, there exists a neighborhood \( U \) of \( E_0 \) such that for \( (S, I, V) \in \text{Int} \Gamma \), the expressions \( S(t) - \frac{2\alpha_1}{\beta_1(1 + R_0)} \) and \( \frac{\alpha_1}{\beta_1} - V(t) - \frac{1}{2} \frac{1}{R_0} + 1 \frac{\alpha_1}{\beta_1} \) are positive. We have that \( \frac{dL_1(t)}{dt} > 0 \) in the set \( U \{ E_0 \} - \{ E_0 \} \).

Now the level sets of \( L_1 \) are the plane

\[
\frac{\beta_1(1 + R_0)}{2k\lambda_1\alpha_1} I(t) + V(t) = C
\]

which move away from the \( S \)-axis when \( C \) increases. Since \( L_2 \) is increasing along the orbits starting in \( \text{Int} \Gamma \), all solutions of system (2.2) move away from \( E_0 \).

**3.5 Theorem**

When \( R_0 > 1 \), the endemic equilibrium \( E \) is globally asymptotically stable.

**Proof:** The system (2.2) is uniformly persistent, and \( E \) is locally asymptotically stable for \( R_0 > 1 \). To prove this theorem, we prove that the system (2.2) has the property of stability of periodic orbits.

Let \( P(t) = (S(t), I(t), V(t)) \) be a periodic solution of system (2.2). To prove the stability of periodic orbits, it is sufficient to prove that the following linear non-autonomous system,

\[
W'(t) = (J^{(2)}P(t))W(t)
\]

is asymptotically stable. The second additive compound matrix is given by

\[
1 + aV + bV^2
\]

For the solution \( P(t) \), we have,

\[
W_1'(t) = -\left( \frac{k\lambda V}{1 + aV + bV^2} + 2\beta I + \gamma \right) W_1(t) + \frac{k\lambda S}{1 + aV + bV^2} W_2(t)
+ \frac{k\lambda S}{1 + aV + bV^2} W_1(t)
\]

\[
W_2'(t) = \left( \frac{a_1}{\beta_1} - V(t) \right) W_2(t) - \left( \frac{k\lambda V}{1 + aV + bV^2} + \beta I + \alpha \right) W_3(t)
+ \frac{k\lambda V}{1 + aV + bV^2} W_1(t)
\]

To prove that system (3.5) is asymptotically stable, we consider the following function

\[
L_2(W_1(t), W_2(t), W_3(t), I(t), V(t), V(t)) = \left\| W_1(t) + \frac{I(t)}{V(t)} W_2(t) + \frac{I(t)}{V(t)} W_3(t) \right\|
\]

Where \( \| \cdot \| \) is the norm in \( \mathbb{R}^3 \) defined by

\[
\left\| W_1(t), W_2(t), W_3(t) \right\| = \sup \left\{ \left| W_1(t) \right|, \left| W_2(t) \right|, \left| W_3(t) \right| \right\}
\]

From theorem (3.4), we obtain that the orbit of \( P(t) \) remains at a positive distance from the boundary of \( \Gamma \). There exists constant \( c > 0 \) such that

\[
L_2(W_1(t), W_2(t), W_3(t), I(t), V(t), V(t)) \geq \left\| W_1(t), W_2(t), W_3(t) \right\| (3.6)
\]
Let \( \{W_1(t), W_2(t), W_3(t)\} \) be a solution of the system (3.5) and
\[
L_1(t) = \sup \left\{ |W_1(t)| \frac{I(t)}{V(t)} |W_1(t) + W_2(t)| \right\} \quad (3.7)
\]
Thus, we obtain the following inequalities
\[
D_1 |W_1(t)| e^{-\left(\frac{k\lambda_1 V}{1+aV+bV^2} + 2\beta_1 + \gamma\right) t} |W_1(t)| e^{-\left(\frac{k\lambda_2 V(1-bV)}{1+aV+bV^2} \frac{V}{I} \right)} + |W_1(t)| + W_2(t) | \leq \lambda_2 \left( \frac{\alpha_2}{\beta_2} - V \right) \frac{I}{V} \left( \frac{I}{V} - \beta_1 \right)
\]
\[
D_1 |W_1(t)| e^{-\left(\frac{k\lambda_1 V}{1+aV+bV^2} + 2\beta_1 + \gamma\right) t} |W_1(t)| e^{-\left(\frac{k\lambda_2 V(1-bV)}{1+aV+bV^2} \frac{V}{I} \right)} + |W_1(t)| + W_2(t) | \leq \lambda_2 \left( \frac{\alpha_2}{\beta_2} - V \right) \frac{I}{V} \left( \frac{I}{V} - \beta_1 \right) |W_1(t)| + W_2(t) | \leq \lambda_2 \left( \frac{\alpha_2}{\beta_2} - V \right) \frac{I}{V} \left( \frac{I}{V} - \beta_1 \right) \quad (3.8)
\]
Thus we obtain
\[
D_1 |W_1(t)| e^{-\left(\frac{k\lambda_1 V}{1+aV+bV^2} + 2\beta_1 + \gamma\right) t} |W_1(t)| e^{-\left(\frac{k\lambda_2 V(1-bV)}{1+aV+bV^2} \frac{V}{I} \right)} + |W_1(t)| + W_2(t) | \leq \lambda_2 \left( \frac{\alpha_2}{\beta_2} - V \right) \frac{I}{V} \left( \frac{I}{V} - \beta_1 \right) \quad (3.9)
\]
From the first equation of system (3.8) and equation (3.9), we get
\[
D_1 L_1(t) \leq \sup \left\{ g_1(t), g_2(t) \right\} W(t)
\]
Where
\[
g_1(t) = -\left( \frac{k\lambda_1 V}{1+aV+bV^2} + 2\beta_1 + \gamma \right) + \frac{k\lambda_2 V(1-bV)}{1+aV+bV^2} \frac{V}{I}
\]
\[
g_2(t) = 2\beta_1 + \gamma\frac{I'}{I} - \beta_1 - \lambda_2 I - \beta_2
\]
Rewriting the second and third equation of system (2.2) as follows
\[
\frac{I'}{I} = -\frac{k\lambda_2 SV}{1+aV+bV^2} - (\gamma + \beta_1)
\]
\[
\frac{V'}{V} = \lambda_2 \left( \frac{\alpha_2}{\beta_2} - (V(t) - \beta_2)
\]
We have
\[
g_1(t) \leq -\left( \frac{k\lambda_1 V}{1+aV+bV^2} + 2\beta_1 + \gamma \right) + \frac{k\lambda_2 S}{1+aV+bV^2} \frac{V}{I}
\]
\[
g_2(t) \leq -\beta_1 + \frac{I'}{I}
\]
\[
g_2(t) \leq -\beta_1 + \frac{I'}{I}
\]
Hence
\[
\sup \left\{ g_1(t), g_2(t) \right\} \leq -\beta_1 + \frac{I'}{I}
\]
From equation (3.10) and Gronwall’s inequality\cite{4}, we obtain
\[
L_1(t) \leq L_1(0) e^{-\beta_1 t} < L_1(0) \frac{\alpha_2}{\beta_2} e^{-\beta_1 t}
\]
which implies that \( L_1(t) \to 0 \) as \( t \to \infty \). By (3.6), we obtain
\[
(W_1(t), W_2(t), W_3(t)) \to 0 \quad \text{as} \quad t \to \infty.
\]
Hence the linear system (3.5) is asymptotically stable and hence the periodic solution is asymptotically orbitally stable. Therefore, the endemic equilibrium \( E^* \) is globally asymptotically stable\cite{8,9}.

4. Numerical Analysis

We present the numerical simulation using MATLAB, to validate the theoretical results. Figure 1 shows that the disease free equilibrium exists for \( R_0 < 1 \). Figure 2 indicates the disease becomes endemic for \( R_0 > 1 \). As the parameters \( a \) and \( b \) increase, number of infective individuals decreases. This is shown in Figure 3 and Figure 4, respectively.
5. Conclusion

We have proposed a vector-host epidemiological model and have studied the dynamical behavior of the model. We established the global asymptotic stability of the disease free equilibrium for $R_0 \leq 1$ and proved that if $R_0 > 1$ the disease becomes endemic. The expression (2.3) shows that $R_0$ does not depend on $a$ and $b$ but it is clear from numerical analysis that the number of infective hosts in endemic state $I$ decreases as $a$ and $b$ increase. It can be found that from expression (2.6) that the number of infective vectors in endemic state $V^*$ approaches zero as $b$ tends to infinity. The results show that if people are aware of preventive measures such as vaccinations and use of bed nets etc, then the spread of disease can be controlled among the host population. Also, awareness of clean surroundings and use of insecticides helps to reduce the number of vectors and eventually eradicate the disease.

6. References