On a Generalization of α-skew McCoy Rings

Mohammad Vahdani Mehrabadi and Shervin Sahebi
Department of Mathematics, Islamic Azad University, Central Tehran Branch, 13185/768, Iran; sahebi@iauctb.ac.ir

Abstract

Objective: To generalize the α-skew McCoy rings. Methods: For a ring endomorphism α, we call a ring R Central α-skew McCoy if for each pair of nonzero polynomials \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; α] \) satisfy \( f(x)g(x) = 0 \), then there exists a nonzero element \( r \in R \) with \( α^r(a)(r) \in C(R) \). Findings: For a ring R, we show that if \( α(e) = e \) for each idempotent \( e \in R \), then \( R \) is Central α-skew McCoy if and only if \( e \) is Central α-skew McCoy if and only if \( 1 - e \) is Central α-skew McCoy. Also, we prove that if \( α^t = I_1 \) for some positive integer \( t \), \( R \) is Central α-skew McCoy if and only if the polynomial ring \( R[x] \) is Central α-skew McCoy if and only if the Laurent polynomial ring \( R[x, x^-1] \) is Central α-skew McCoy. Moreover, we give some examples to show that if \( R \) is Central α-skew McCoy, then \( T_α(R) \) is not necessarily Central α-skew McCoy, but \( D_\alpha(R) \) and \( V_\alpha(R) \) are Central α-skew McCoy, where \( D_\alpha(R) \) and \( V_\alpha(R) \) are the subrings of the triangular matrices with constant main diagonal and constant main diagonals, respectively.

Keywords: Central α-skew McCoy Ring, McCoy Ring, Ore Extension, skew McCoy ring, Triangular Matrix Ring, Skew Polynomial Ring

1. Introduction

Throughout this paper, \( R \) denotes an associative ring with identity and \( α \) is a ring endomorphism. We denote \( R[x; α] \) the Ore extension whose elements are the polynomials \( \sum_{i=0}^{n} a_i x^i, a_i \in R \), where the addition is defined as usual and the multiplication subject to the relation \( xa = α(a)x \) for any \( a \in R \). For notation \( T_α(R), S_α(R), R[x], C[R] \) and \( e \), denote, its upper triangular matrix ring, its diagonal matrix ring, polynomial ring over \( R \), the center of a ring \( R \) and the matrix with \((i, j)\)-entry 1 and elsewhere 0, respectively. A ring \( R \) is Armendariz if whenever polynomials \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x] \) satisfy \( f(x)g(x) = 0 \), then \( ab = 0 \) for all \( i \) and \( j \). Agayev called a ring \( R \) Central Armendariz if whenever polynomials \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x] \) satisfy \( f(x)g(x) = 0 \), then \( ab = 0 \) for all \( i \) and \( j \). They showed that the class of central Armendariz rings lies precisely between classes of Armendariz rings and abelian rings (that is, its idempotents belong to \( C(R) \)). Let \( α \) be an endomorphism of a ring \( R \). According to \( α \), a ring \( R \) is called α-skew Armendariz if \( f(x)g(x) = 0 \), such that \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; α] \) implies that \( α^r(a)(b) = 0 \) for all \( i, j \). Rege and Chhawchharia called a noncommutative ring \( R \) right McCoy if whenever nonzero polynomials \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x] \) satisfy \( f(x)g(x) = 0 \), there exists nonzero elements \( r \in R \) such that \( ar = 0 \). Left McCoy rings are defined similarly. A ring is McCoy if it is both left and right McCoy. Clearly, Armendariz rings are McCoy. So far McCoy rings are generalized in several forms. In Basser, Kwak and Lee called a ring \( R \), α-skew McCoy ring with respect to \( α \) if for any nonzero polynomials \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; α] \) satisfy \( f(x)g(x) = 0 \), there exists nonzero elements \( r \in R \) such that \( ar = 0 \).

Motivated by the above results, for an endomorphism \( α \) of a ring \( R \), we call a ring \( R \) Central α-skew McCoy if for each pair of nonzero polynomials \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; α] \) satisfy \( f(x)g(x) = 0 \), then there exists a nonzero element \( r \in R \) with \( α^r(a)(r) \in C(R) \). Clearly, all commutative rings, α-skew McCoy rings and α-skew Armendariz rings are Central α-skew McCoy.
2. Central a-skew McCoy Rings

We start this section by the following definition:

**Definition 2.1.** Let \( a \) be an endomorphism of a ring \( R \). The ring \( R \) is called a Central \( a \)-skew McCoy ring if for each pair of nonzero polynomials \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; a] \) satisfy \( f(x)g(x) = 0 \), then there exists a nonzero element \( r \in R \) with \( a f(r) \in C(R) \).

It is clear that a \( a \)-skew McCoy rings are Central \( a \)-skew McCoy, but the converse is not always true by the following example.

**Example 2.2.** Let \( R_1 \) and \( R_2 \) be any commutative rings. Suppose \( R = R_1 \oplus R_2 \) with the usual addition and multiplication. Let \( a : R \rightarrow R \) be an endomorphism defined by \( a(a_i) + b(a_j) = (a_i + b, a_j) \), then for \( f(x) = (1,0) - (1,0)x \) and \( g(x) = (0,1) + (0,1)x \) in \( R[x; a] \), \( f(x)g(x) = 0 \), but if there exists \( (r_1, r_2) \in R \) such that \( (0,1)(r_1, r_2) = 0 \) and \( (0,1)(x_1, x_2) = 0 \), then \( r_1 = r_2 = 0 \). Therefore \( R \) is not a Central \( a \)-skew McCoy. But \( R \) is Central \( a \)-skew McCoy, since \( R \) is commutative.

**Example 2.3.** Let \( K \) be a field and \( K(x, y, z) \) be the free algebra with noncommuting indeterminates \( x, y, z \) over \( K \). Let \( B \) be the factor ring of \( K(x, y, z) \) with relations

\[
x^2 = yx = x, \quad y^2 = xy = y, \quad z^2 = 0, \quad zx = xz = yz = z
\]

We coincide \( x, y, z \) with their images in \( B \), for simplicity. Consider the subring of \( R \) generated by \( \{a, x, y, z|a \in K\} \), say \( S \). Then every element of \( S \) is of the form, \( a + a_i x + a_y + a_z \) with \( a \)'s in the field \( K \). Let \( a : S \rightarrow S \) be an endomorphism defined by \( a(a_i + a_j x + a_y + a_z) = a_i + a_j x + a_y + a_z \). By the construction of \( S \), we have \( (x + x')((1 - x') + (1 - y)t) = 0 \) while \( x + yt \) and \( 1 - x' + (1 - y)t \) are nonzero polynomials over \( S \). Assume there exists \( a_i + a_j x + a_y + a_z \in R \) such that \( x(a_i + a_j x + a_y + a_z) = 0 \) and \( y^2(a_i + a_j x) = 0 \). Then \( a_i = a_j = a_y = a_z = 0 \). Thus \( S \) is not \( a \)-skew McCoy.

Next we show that \( S \) is Central \( a \)-skew McCoy. Let

\[
f(t) = \sum_{i=0}^{n} (a_{i_1} + a_{i_2} x + a_{i_3} y + a_{i_4} z)t^i \quad \text{and} \quad g(t) = \sum_{j=0}^{m} (b_{j_1} + b_{j_2} x + b_{j_3} y + b_{j_4} z)t^j
\]

be nonzero in \( S[t] \) with \( f(t)g(t) = 0 \). Then

\[
(a_{i_1} + a_{i_2} x + a_{i_3} y + a_{i_4} z)z(a_{i_5} + a_{i_6} + a_{i_7} z) \in C(S)
\]

since

\[
(a_{i_1} + a_{i_2} + a_{i_3})z(a_{i_4} x + a_{i_5} y + a_{i_6} z) = (a_{i_1} + a_{i_2} + a_{i_3})z(a_{i_4} + a_{i_5} + a_{i_6} z)
\]

for each \( a_{i_1} + a_{i_2} x + a_{i_3} y + a_{i_4} z \in S \).

Now we turn our attention to study some extensions of Central \( a \)-skew McCoy rings.

Let \( R \) be a ring, where \( k \in Z, a_k \) an endomorphism of \( R_k \). Let \( R = \prod_{k \in Z} R_k \) and \( \langle \oplus_{k \in Z} R_k | 1 \rangle \) be the subring of \( R \) generated by \( \oplus_{k \in Z} R_k \) and \( \{1_k\} \). Then the map \( a : R \rightarrow R \) defined by \( a(a_k) = (a_k) \) is an endomorphism of \( R \).

**Proposition 2.4.** Let \( R \) be a ring with an endomorphism \( a_k \), where \( k \in Z \). Then \( R \) is Central \( a_k \)-skew McCoy for each \( k \in Z \) if and only if \( R = \prod_{k \in Z} R_k \) is Central \( a_k \)-skew McCoy if and only if \( \langle \oplus_{k \in Z} R_k | 1 \rangle \) is Central \( a_k \)-skew McCoy.

Proof. Let each \( R_k \) be Central \( a_k \)-skew McCoy and \( f(x) = \sum_{i=0}^{n} a_i x^i \), \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; a] \) \( \setminus \{0\} \) such that \( f(x)g(x) = 0 \), where \( a_i = (a_i(k)), b_j = (b_j(k)) \). If there exists \( t \in Z \) such that \( a_i(t) = 0 \) for each \( 0 \leq t \leq m \), then we have \( a f(r) \in C(R) \) where \( c = (0,0,1,0,0,0) \). Now, suppose for each \( k \in Z \), there exists \( 0 \leq t_k \leq m \) such that \( a_k(t_k) \neq 0 \). Since \( g(x) \neq 0 \), there exists \( t \in Z \) and \( 0 \leq t \leq j \) such that \( b^{(j)}(t) \neq 0 \). Consider \( f_t(x) = \sum_{i=0}^{m} a^{(i)}(x) x^i \) and \( g_t(x) = \sum_{j=0}^{n} b^{(j)}(x) x^j \in R[x; a] \setminus \{0\} \). We have \( f_t(x)g_t(x) = 0 \).

Thus there exists nonzero \( c \in R \) such that \( a^{(i)}(x) \in C(R) \), for each \( 0 \leq i \leq m \), since \( R \) is Central \( a_k \)-skew McCoy ring. Therefore \( a^{(i)}(x) \in C(R) \), for each \( 0 \leq t \leq m \), where \( c = (0, \ldots, 0, c, 0, \ldots, 0) \) and so \( R \) is Central \( a_k \)-skew McCoy. Conversely, suppose \( R \) is Central \( a_k \)-skew McCoy and \( t \in Z \). Let \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_j x^j \) be nonzero polynomials in \( R[x; a] \) such that \( f(x)g(x) = 0 \). Let

\[
F(x) = \sum_{j=0}^{m} (0,0,0,0,0,0)x^i,
\]

\[
G(x) = \sum_{j=0}^{m} (0,0,0,0,0,0)x^i \in R[x; a]
\]

Hence \( F(x)G(x) = 0 \) and so there exists \( 0 \neq c = (c_1, c_2, \ldots, c_{t-1}, c_t, c_{t+1}, \ldots, c_{m-1}, c_m) \) such that \( (0,0,0,0,0,0) \in C(R) \). Therefore, \( a^{(i)}(x) \in C(R) \) and so \( R \) is Central \( a_k \)-skew McCoy. By a similar way, one can prove that \( \langle \oplus_{k \in Z} R_k | 1 \rangle \) is Central \( a_k \)-skew McCoy if and only if each \( R_k \) is Central \( a_k \)-skew McCoy.

**Corollary 2.5.** Let \( D \) be a ring and \( C \) a subring of \( D \) with \( 1_D \in C \). Let

\[
R(C,D) = \left\{ (d_1, \ldots, d_n, c, c, \ldots) \mid d_i \in D, c \in C, n \geq 1 \right\}
\]

with addition and multiplication defined component-wise, \( R(C, D) \) is a ring. Let \( a \) be an endomorphism of
such that \(a(C) \subseteq C\). Then the map \(\overline{a} : R \to R\) defined by \(\overline{a}((a(D), a(D)) = (a(C), a(D))\) is an endomorphism of \(R\). Then \(D\) is Central \(\alpha\)-skew McCoy if and only if \(R(D, C)\) is Central \(\overline{\alpha}\)-skew McCoy.

**Proposition 2.6.** Let \(\alpha\) be an endomorphism of a ring \(R\). Let \(S\) be a ring and \(\varphi : R \to S\) an isomorphism. Then \(R\) is central \(\alpha\)-skew McCoy if and only if \(S\) is central \(\varphi \alpha \varphi^{-1}\)-skew McCoy.

Proof. Let \(\alpha' = \varphi \alpha \varphi^{-1}\). Clearly, \(\alpha'\) is an endomorphism of \(S\). Suppose that \(\alpha' = \varphi(a)\), for \(a \in R\). Note that \(\varphi(\alpha'(b)) = \alpha' \varphi(b)\) for all \(a, b \in R\). Also and

\[
f(x) = \sum_{i=0}^{n} a_i x^i, \quad g(x) = \sum_{i=0}^{m} b_i x^i
\]

be nonzero polynomials in \(R[x; a]\) if and only if \(f'(x) = \sum_{i=0}^{n} a_i x^i\), \(g'(x) = \sum_{i=0}^{m} b_i x^i\) be nonzero polynomials in \(S[x; a']\).

On the other hand \(f(x)g(x) = 0\) in \(R[x; a]\) if and only if \(f'(x)g'(x) = 0\) in \(S[x; a']\). Also since \(\varphi\) is an isomorphism,

\[
a'i'(\alpha(b)) = \alpha'(\varphi(b)) = \varphi(\alpha'(b)) = \varphi(\alpha(b)) = \varphi(a'(b)) \in C(S)
\]

so that \(a\alpha'(c) \in C(S)\). Thus \(R\) is Central \(\alpha\)-skew McCoy if and only if \(S\) is Central \(\varphi \alpha \varphi^{-1}\)-skew McCoy.

The following example shows that, if \(R\) is Central \(\alpha\)-skew McCoy, then \(T_2(R)\) is not necessary Central \(\overline{\alpha}\)-skew McCoy.

**Example 2.7.** Let \(R\) be a commutative ring and \(\overline{a} : T_2(R) \to T_2(R)\) be an endomorphism defined by \(\overline{a}(e_{11} + P_1 e_{12} + P_2 e_{22}) = e_{11} + P_1 e_{12} + P_2 e_{22}\). Let \(f(x) = e_{11} + (e_{12} + e_{21})x, g(x) = -e_{22} + (e_{12} + e_{21})x \in (T_2(R))[x; a]\). Then \(f(x)g(x) = 0\). If \(T_2(R)\) is central \(\overline{\alpha}\)-skew McCoy, then there exists \(P = P_1 e_{11} + P_2 e_{12} + P_2 e_{22} \in T_2(R)\) such that \(e_{11} P, (e_{12} + e_{21}) \alpha(P) \in C(T_2(R))\). Therefore \(P_1 e_{11} + P_2 e_{12}\) and \(P_2 e_{22}\) is not Central \(\overline{\alpha}\)-skew McCoy. But \(R\) is Central \(\alpha\)-skew McCoy (for any endomorphism \(a : R \to R\), since \(R\) is commutative.

Let \(\alpha\) be an endomorphism of a ring \(R\). The endomorphism \(a\) of \(R\) is extended to the endomorphism \(\overline{a} : T_n(R) \to T_n(R)\) defined by \(\overline{a}(a_j) = (a(a_j))\).

**Theorem 2.8.** Let \(\alpha\) be an endomorphism of a ring \(R\). Then \(R\) is Central \(\alpha\)-skew McCoy if and only if one of the following holds:

1. \(D_n(R) = \Bigg\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ a & a_{23} & a_{33} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \Bigg\} \quad a, a_j \in R\)

is Central \(\overline{\alpha}\)-skew McCoy for any \(n \geq 1\).

2. \(V_n(R) = \Bigg\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_2 \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \Bigg\} \quad a_1, a_2, \cdots, a_n \in R\)

\(a_n \in R\) and \(x(x^n) = \sum x^i \in D_n(R)\) is Central \(\overline{\alpha}\)-skew McCoy for any \(n \geq 1\), where \((x^n)\) is a two-sided ideal of \(R[x; a]\) generated by \(x^n\).

Proof. (1) Let \(F(x) = \sum A_i x^i\), \(G(x) = \sum B_j x^j \in D_n(R)[x; a]\) where,

\[
\begin{pmatrix} a_i & a_{12} & \cdots & a_{in} \\ 0 & a_i & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_i \end{pmatrix}, \quad \begin{pmatrix} b_j & b_{12} & \cdots & b_{jn} \\ 0 & b_j & \cdots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_j \end{pmatrix}
\]

Then

\[
\begin{pmatrix} f(x) & f_{12}(x) & \cdots & f_{in}(x) \\ 0 & f(x) & \cdots & f_{1n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(x) \end{pmatrix}, \quad \begin{pmatrix} g(x) & g_{12}(x) & \cdots & g_{in}(x) \\ 0 & g(x) & \cdots & g_{1n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g(x) \end{pmatrix}
\]

Where \(f(x) = \sum a_i x^i, f_{12}(x) = \sum a_i x^i, g(x) = \sum b_j x^j\) for any \(k = 1, 2, \ldots, n, l = 2, 3, \ldots, n\) and \(k < l\). Suppose \(F(x)G(x) = 0, F(x)G(x) \neq 0\). Set \(H(x) = F(x)G(x) = (h_l(x))\) for \(p, q = 1, 2, \ldots, n\).

CASE 1. If \(f(x) \neq 0, g(x) \neq 0\), then \(h_{11}(x) = f(x)g(x) = 0\). Since \(R\) is Central \(\alpha\)-skew McCoy there exists \(r \in R[0]\) such that \(a\alpha'(r) \in C(R)\). Let \(A = rE_{11}\). Then \(A\overline{\alpha}(A) \in C(D_n(R))\).

CASE 2. If \(f(x) \neq 0\) and \(g(x) \neq 0\), then there exists \(h_{11}(x) \neq 0\), such that \(g_{11}(x) = 0\) for some \(k, l\) and \(1 \leq u \leq n - k, \) since \(G(x) \neq 0\). So \(h_{11}(x) = f(x)g_{11}(x) = 0\). Hence there
exists \( r \in R\setminus\{0\} \) such that \( a_\ell r(r) \in C(R) \). Let \( A = rE_{1n} \). Then \( \Lambda_1 A(\Lambda_1) \in C(D_n(R)) \).

CASE 3. If \( f(x) = 0 \), then clearly \( A_1 \Lambda_1(A) = 0 \), where \( A = E_{1n} \).

Thus \( D_n(R) \) is Central \( \Lambda - \) skew McCoy.

Conversely, assume that \( f(x) g(x) = 0 \), where

\[
f(x) = \sum_{i=0}^{n} a_i x^i, \quad g(x) = \sum_{j=0}^{m} b_j x^j \]

are nonzero polynomials of \( R[x] \). Let \( F(x) = \sum_{i=0}^{n} A_i x^i, \quad G(x) = \sum_{j=0}^{m} B_j x^j, \)

where

\[
A_i = \begin{pmatrix}
a_i & a_i & \cdots & a_i \\
a_i & a_i & \cdots & a_i \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_i
\end{pmatrix}, \quad B_j = \begin{pmatrix}
b_j & b_j & \cdots & b_j \\
0 & b_j & \cdots & b_j \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b_j
\end{pmatrix}
\]

for any \( i = 0, 1, 2, \ldots, n \), \( j = 0, 1, 2, \ldots, m \). Then

\[
F(x)G(x) = \begin{pmatrix}
f(x) & f(x) & \cdots & f(x) \\
f(x) & f(x) & \cdots & f(x) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & f(x)
\end{pmatrix} \begin{pmatrix}
g(x) & g(x) & \cdots & g(x) \\
g(x) & g(x) & \cdots & g(x) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & g(x)
\end{pmatrix}
\]

Hence there exists \( A = \begin{pmatrix}
s & s_{12} & \cdots & s_{1n} \\
0 & s & \cdots & s_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s
\end{pmatrix} \in D_n(R) \setminus \{0\} \)

such that \( A_1 \Lambda_1(A) \in C(D_n(R)) \), since \( D_n(R) \) is Central \( \Lambda - \) skew McCoy. If \( S \neq 0 \), then \( a_\ell(s) \in C(R) \). If \( S = 0 \), then there exists \( S_p \neq 0 \) for some \( i, j \), such that \( S_{i(j)} = 0 \) for any \( 1 \leq n - i \). We also have \( a_\ell(s_p) \in C(R) \). Thus, \( R \) is Central \( \ell - \) skew McCoy.

(2) The proof is similar to (1).

**Theorem 2.8.** Let \( a \) be an endomorphism of a ring \( R \) and \( a' = I_x \) for some positive integer \( t \). Then \( R \) is Central \( a - \) skew McCoy if and only if \( R[x] \) is Central \( a - \) skew McCoy.

Proof. Assume that \( R \) is Central \( a - \) skew McCoy. Let \( R[x][y] \) denote the polynomial ring with an indeterminate \( y \) over \( R[x] \). Suppose that \( p(y) = f_0 + f_1 y + \cdots + f_m y^n \), \( q(y) = g_0 + g_1 y + \cdots + g_{n'} y^m \in R[x][y] \) and \( p(y) q(y) = 0 \). We also let \( f_i = a_{i0} + a_{i1} x + \cdots + a_{in} x^i \) and \( g_j = b_{j0} + b_{j1} x + \cdots + b_{jn} x^j \) for each \( 0 \leq i \leq m \) and \( 0 \leq j \leq n' \), where \( a_{i0}, \ldots, a_{in}, b_{j0}, \ldots, b_{jn} \in R \). We claim that there exists \( c \in R[x] \) such that \( f_\ell(c) \in C(R[x]) \), for all \( 0 \leq i \leq m \), \( 0 \leq j \leq n \). Take a positive integer \( k \) such that \( k \geq \deg(f_\ell(x)) + \cdots + \deg(f_m(x)) + \deg(g(x)) + \cdots + \deg(g(x)) \), where the degree is as polynomials in \( R[x] \) and the degree of zero polynomial is taken to be 0. Since \( p(y) q(y) = 0 \) in \( R[x][y] \), we have

\[
\begin{align*}
f_0(x) g_0(x) &= 0 \\
f_0(x) g_1(x) + f_1(x) a_\ell(g_0(x)) &= 0 \\
& \vdots \\
f_m(x) a^n(g_n(x)) &= 0
\end{align*}
\]

in \( R[x] \). Now put

\[
f(x) g(x) = f_0(x) g_0(x) + (f_0(x) g_1(x) + f_1(x) a_\ell(g_0(x))) x^{2k+1} + \cdots + f_m(x) a^n(g_n(x)) x^{mk+l+n}.
\]

Note that \( a' = I_x \). Then

\[
f(x) g(x) = f_0(x) g_0(x) + (f_0(x) g_1(x) + f_1(x) a_\ell(g_0(x))) x^{2k+1} + \cdots + f_m(x) a^n(g_n(x)) x^{mk+l+n} \]

in \( R[x] \). Using (1) and (2), we have \( f(x) g(x) = 0 \) in \( R[x] \). In the other hand, from (2) we have

\[
f(x) g(x) = (a_{i0} + a_{i1} x + \cdots + a_{in} x^i + a_{i1} x^{i+1} + a_{i2} x^{i+2} + \cdots + a_{in} x^{i+n})
\]

\[
+ \cdots + a_{m0} x^{m0} x^{m0} + a_{m1} x^{m1} x^{m1} + \cdots + a_{mn} x^{mn} x^{mn} x^{mn} + a_{m1} x^{m1} x^{m1} + \cdots + a_{mn} x^{mn} x^{mn} x^{mn}
\]

\[
= 0.
\]

Since \( R \) is Central \( a - \) skew McCoy and \( a' = I_x \), so there exists \( c \in R \) such that \( a_\ell(c) \in C(R) \) for each \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \). Since \( C(R) \) is closed under addition, we have \( f(x) a'(c) \in C(R[x]) \) for every \( 0 \leq i \leq m \). Now it is easy to see that \( f(x) a'(c) \in C(R[x]) \), and hence \( R[x] \) is Central \( a - \) skew McCoy.

Conversely, suppose that \( R[x] \) is Central \( a - \) skew McCoy. Let \( f(x) g(x) = 0 \) for nonzero polynomial \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x] \). Suppose \( f(t) \) and \( g(t) \) is constant polynomial in \( R[x][y] \) such that \( f(t) g(t) = 0 \). Then there exists \( 0 \neq c(x) = c_{00} + c_{01} x + \cdots + c_{m0} x^m \in R[x] \) such that \( f(t) a'(c(x)) \in C(R[x]) \). So \( a_\ell(c(x)) \in C(R) \). Since \( c(x) \) is a nonzero element, then at least one of the \( c_\ell 's \) is nonzero element, for example \( c_{00} \). So \( a_\ell(c_{00}) \in C(R) \).

Therefore \( R \) is Central \( a - \) skew McCoy.
Proposition 2.10. Let $R$ be a ring and $e$ a central idempotent element of $R$ and $a$ be an endomorphism of a ring $R$ with $a(e) = e$. Then $R$ is Central $a$-skew McCoy ring if and only if $eR$ is Central $a$-skew McCoy ring if and only if $(1 - e)R$ is Central $a$-skew McCoy ring.

Proof. Assume that $R$ is Central $a$-skew McCoy ring and consider $f(x) = \sum_{i=0}^{n} e a_i x^i$, $g(x) = \sum_{j=0}^{m} e b_j x^j$ $\in eR[x; a] \setminus \{0\}$ such that $f(x)g(x) = 0$. Since $R$ is Central $a$-skew McCoy ring, there exists $s \in R$ such that $(e a_i a'(s)) \in C(R)$. So $(e a_i a'(s)) r = r(e a_i a'(s))$, for any $r \in R$. Therefore $((e a_i a'(es)))(er) = (er)((e a_i a'(es)))$. So $(e a_i a'(es)) \in C(eR)$. Hence $eR$ is Central $a$-skew McCoy ring. Similarly, we prove that $(1 - e)R$ is Central $a$-skew McCoy ring.

Conversely, assume that $eR$ is Central $a$-skew McCoy ring. Consider $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x; a] \setminus \{0\}$ such that $f(x)g(x) = 0$. Clearly, $ef(x), eg(x) \in eR[x; a]$ and $(ef(x))(eg(x)) = e^2 f(x)g(x) = ef(x)g(x) = 0$, since $e$ is a central idempotent element of $R$. Then there exists $s \in R$ such that $(e a_i a'(s)) \in C(eR)$. So $(e a_i a'(s)) r = r(e a_i a'(s))$. Hence, $R$ is Central $a$-skew McCoy ring. Similarly, this fact is satisfied if $(1 - e)R$ is Central $a$-skew McCoy ring.

Let $S$ denote a multiplicatively closed subset of a ring $R$ consisting of central regular elements. Let $RS^{-1}$ be the localization of $R$ at $S$. Then we have:

Proposition 2.11. Let $a$ be an automorphism of a ring $R$. If $R$ is Central $a$-skew McCoy, then $RS^{-1}$ is Central $a$-skew McCoy ring.

Proof. Suppose that $R$ is Central $a$-skew McCoy ring. Let $f(x) = \sum_{i=0}^{n} (a_i/s_i) x^i$, $g(x) = \sum_{j=0}^{m} (b_j/d_j) x^j \in RS^{-1}[x; a]$ and $f(x)g(x) = 0$. Let $a_i s_i^{-1} = c^{-1} a'_i$ and $b_j d_j^{-1} = d^{-1} b'_j$ with $c, d$ regular elements in $R$. Then we have $(a'_0 + \cdots + a'_n x^n)d^{-1}(b'_0 + \cdots + b'_m x^m) = 0$. We know that for each element $f(x) \in RS^{-1}[x; a]$ there exists a regular element $C \in R$ such that $f(x) = h(x)c^{-1}$, for some $h(x) \in R[x; a]$, or equivalently, $f(x)c \in R[x; a]$. Therefore there exist a regular element $e$ in $R$ and $(b''_0 + \cdots + b''_m x^m) \in R[x; a]$, such that $d^{-1}(b'_0 x^0 + \cdots + b'_n x^n) = (b''_0 + \cdots + b''_m x^m)e^{-1}$. Hence $(a'_0 + \cdots + a'_n x^n)(b''_0 + \cdots + b''_m x^m) = 0$. Since $R$ is Central $a$-skew McCoy, then there exists $r \in R$ such that $a'_i a'(r) \in C(R)$ for all $i$. Therefore $a_i s_i^{-1} a'(r)$ are central in $RS^{-1}$ for all $i$.

Corollary 2.12. Let $R$ be a ring and $a$ an automorphism of $R$, such that $a^i = I_R$ for some positive integer $t$, then the following are equivalent:

1. $R$ is Central $a$-skew McCoy.
2. $R[x]$ is Central $a$-skew McCoy.
3. $R[x, x^{-1}]$ is Central $a$-skew McCoy.

Proof. Let $S = \{1, x, x^2, x^3, x^4, \ldots\}$. Then $S$ is a multiplicatively closed subset of $R[x]$ consisting of central regular elements. Then the proof follows from Proposition 2.11.

3. Conclusions

In this paper, we have defined a new class of rings called Central $a$-skew McCoy and illustrated with some examples. Central $a$-skew McCoy rings are proper extension of $a$-skew McCoy rings. Also, some related results have been studied and proved. This particular class of ring is not Morita Invariant.

4. Acknowledgements

This paper is supported by Islamic Azad University Central Tehran Branch (IAUCTB). The authors want to thank the authority of IAUCTB for their support to complete this research.

5. References