In this paper, we prove that the extended duplicate graph of twig is mean labeling.

Keywords: Extended Duplicate Graphs, Graph Labeling, Mean Labeling

1. Introduction

Graph theory is the fast growing area of combinatorics. Graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs serve as useful models for a wide range of applications such as data security, communications networks, X-ray, radar, circuit design and data base management. After the introduction of graph labeling, various labeling of graphs such as graceful labeling, cordial labeling, prime cordial labeling, magic labeling, anti magic labeling etc., have been studied in over 1400 papers. Graphs that are considered in this paper are finite, simple and undirected graph G = (V, E) having p vertices and q edges. Several researchers refer to Rosa’s work.

In Somasundaram and Ponraj introduced the idea of mean labeling of graphs. In their work, they have shown that the graphs P_n, C_n, K_m and K_2 + mK_1 are mean graphs.

We will provide some definitions which are necessary for this paper.

1.1 Twig

A graph G(V, E) obtained from a path by attaching exactly two pendent edges to each internal vertices of the path is called a Twig graph, where m is the internal vertices. In general, a twig T_m has 3m + 2 vertices and 3m + 1 edges.

1.2 Duplicate Graph

Let G (V, E) be a simple graph and the duplicate graph of G is DG = (V_1, E_1), where the vertex set V_1 = V ∪ V’ and V ∩ V’ = φ and f: V → V’ is bijective (for v ∈ V, we write f(v) = v’ for convenience) and the edge set E_1 of DG is defined as the edge ab is in E if and only if both a'b' and a'b are edges in E_1. It is evident that the duplicate graph of path graph is disconnected.

1.3 Extended Duplicate Graph

If G (V, E) is a Twig graph T_m, the duplicate graph of G is DG = (V_1, E_1) where the vertex set V_1 = V ∪ V’ and V ∩ V’ = φ be a duplicate graph of a Twig T_m. By adding an edge between any one vertex from V to any other vertex in V’, except the terminal vertices of V and V’. For convenience, let us take v_i ∈ V and v'_i ∈ V’ and thus the edge (v_i, v'_i) is formed. Call this new graph as the Extended Duplicate Graph of the Twig T_m and it is denoted as EDG (T_m).

1.4 Mean Labeling

For a graph G having p vertices and q edges is called a mean labeling, if there is an injective function f from the vertices of G to {0,1,2, ..., q} such that each edge uv is labeled as \( \frac{f(u) + f(v)}{2} \) if f(u) + f(v) is even and uv labeled as \( \frac{f(u) + f(v) + 1}{2} \) if f(u) + f(v) is odd, then the obtaining edge labels are different.

*Author for correspondence
2. Main Results

2.1 Mean Labeling

In this section, we now present an algorithm and prove the existence of Mean labeling for EDG(Tm).

Algorithm

Procedure (mean labeling for EDG(Tm))

\[ V \leftarrow \{v'_1, v'_2, \ldots, v'_{2m}, v'_{2m+1}, v'_3, v'_4, \ldots, v'_{2m+3}, v'_{2m+4}\} \]

\[ E \leftarrow \{e'_1, e'_2, \ldots, e'_{2m}, e'_{2m+1}, e'_2, e'_3, \ldots, e'_{2m+2}\} \]

if \((m = 2n)\) then

for \(i = 0\) to \((m - 2)/2\) do

for \(j = 0\) to \(2\) do

\[ v'_{2ij} \leftarrow 6i + 2 + 2j; \]

\[ v'_{3ij} \leftarrow 6m + 6i - 2j; \]

\[ v'_{3ij} \leftarrow 6m + 6i + 1 - 2j; \]

\[ v'_{3ij} \leftarrow 6i + 3 + 2j; \]

end for

end for

else

for \(i = 0\) to \((m - 1)/2\) do

for \(j = 0\) to \(2\) do

\[ v'_{2ij} \leftarrow 6i + 2 + 2j; \]

\[ v'_{3ij} \leftarrow 6m + 6i + 1 - 2j; \]

\[ v'_{3ij} \leftarrow 6i + 3 + 2j; \]

end for

end for

for \(i = 0\) to \((m - 3)/2\) do

for \(j = 0\) to \(2\) do

\[ v'_{2ij} \leftarrow 6m - 6i - 2j; \]

\[ v'_{3ij} \leftarrow 6m + 6i + 1 - 2j; \]

\[ v'_{3ij} \leftarrow 6m + 6i + 1 - 2j; \]

end for

end for

end if

Theorem: For a twig \(T_m\), \(m \geq 1\), the extended duplicate graph of is mean labeling.

Proof: Let \(T_m\), \(m \geq 1\) be a twig. In order to label the vertices, define a function \(f: V \rightarrow \{1, 2, \ldots, 6m + 3\}\) as given in algorithm.

Case 1: If \(m = 2n\); \(n \in \mathbb{N}\), the vertices \(v'_1, v'_2, v'_1\) and \(v'_2\) receive label 0, 6m + 2, 6m + 3 and 1 respectively; the vertices \(v'_{3ij}\) receive label \(6i + 2 + 2j\) for \(0 \leq i \leq (m - 2)/2\) and \(0 \leq j \leq 2\); the vertices \(v'_{3ij}\) receive label \(6i - 6i - 2j\); the vertices \(v'_{3ij}\) receive label \(6i + 6i + 1 - 2j\); the vertices \(v'_{3ij}\) receive label \(6i + 3 + 2j\). Hence all the \(6m + 4\) vertices \(v'_1, v'_2, v'_3, \ldots, v'_{2m+1}, v'_{2m+2}\) receive labeled 0, 6m + 2, 4, 6, 6m, 6m - 2, 6m - 4, 8, 10, 12, \ldots, 3m + 6, 3m + 4, 3m + 2 and the vertices \(v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, \ldots, v'_3m, v'_{3m+1}, v'_{3m+2}\) receive consecutive odd numbers labeled 6m+3, 1, 6m + 1, 6m - 1, 6m - 3, 3, 5, 7, \ldots, 3m - 3, 3m - 1 and 3m + 1.

Case 2: If \(m = 2n - 1\); \(n \in \mathbb{N}\), the vertices \(v'_1, v'_2, v'_1\) and \(v'_2\) receive label 0, 6m + 2, 6m + 3 and 1 respectively; the vertices \(v'_{3ij}\) receive label \(6i + 2 + 2j\) and the vertices \(v'_{3ij}\) receive label \(6m + 6i + 1 - 2j\) for \(0 \leq i \leq (m - 1)/2\) and \(0 \leq j \leq 2\); the vertices \(v'_{3ij}\) receive label \(6m - 6i + 2j\) and the vertices \(v'_{3ij}\) receive label \(6i + 3 + 2j\) for \(0 \leq i \leq (m - 3)/2\) and \(0 \leq j \leq 2\). Hence all the \(6m + 4\) vertices are labeled such that the vertices \(v'_1, v'_2, v'_3, \ldots, v'_{2m+1}, v'_{2m+2}\) receive 0; 6m + 2, 4, 6, 6m, 6m - 2, 6m - 4, 8, 10, 12, \ldots, 3m - 1, 3m + 1, 3m + 3 and the vertices \(v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, \ldots, v'_{2m+1}, v'_{2m+2}\) receive labeled 6m + 3, 1, 6m + 1, 6m - 1, 6m - 3, 3, 5, 7, \ldots, 3m + 4, 3m + 2, 3m.

Now we define the induced function \(f^* : E \rightarrow \mathbb{N}\) as follows

\[
\begin{align*}
    f^*(v'_v) = & \begin{cases} 
    \frac{f(v'_v) + f(v'_v)}{2} & \text{if } f(v'_v) + f(v'_v) \text{ is even} \\
    \frac{f(v'_v) + f(v'_v) + 1}{2} & \text{if } f(v'_v) + f(v'_v) \text{ is odd; } v'_v \in V. 
    \end{cases}
\end{align*}
\]

The induced function gives the consecutive numbers 0, 1, 2, ..., 3m - 1, 3m, 3m + 1, 3m + 2 for the edges \(e'_1, e'_2, \ldots, e'_{2m+2}\).

Figure 1. Mean labeling for EDG(T_2) and EDG(T_3).
and the consecutive numbers $3m + 3, 3m + 4, 3m + 5, \ldots, 6m, 6m + 1, 6m + 2, 6m + 3$ for the edges $e'_{3m+1}, e'_{3m+2}, \ldots, e'_{3m}, e'_{3m+1}, e'_{3m+2}$ and $e'$. Hence all the $6m + 3$ edges are $1, 2, 3 \ldots, 6m, 6m + 1, 6m + 2, 6m + 3$, which are all different.

Thus, the extended duplicate graph of $T_m$, $m \geq 1$, is mean labeling.

**Example:** Mean labeling for graphs EDG(T$_2$) and EDG(T$_3$) are shown in Figures 1(a) and 1(b) respectively.

### 3. References