Fixed Point Result for $\alpha P_{f,g}$–Integral Contractive Mappings with Applications

Mecheraoui Rachid
Department of Mathematics and Informatics, Abbes Laghour University-Khenchela-Algeria; rachid.mecheraoui@yahoo.fr

Abstract

Background/Objectives: In this paper, we introduced the concept of $\alpha P_{f,g}$ integral contractive mappings, which is a new class of integral contractive mappings and using this notion we establish a new fixed point theorem. Findings: Our paper represents a generalization and extension of fixed point theorems for mappings satisfying contractive conditions of integral type where the contractive inequality depends on rational and irrational expression. In particular, we omitted the condition of continuity (which is a very strong condition and appear in almost all papers using contractive mapping of rational type) from many existing results. Application/Improvements: As a direct consequence, some new results of integral type for rational and irrational contraction maps are presented to illustrate our obtained result.

Keywords: Fixed Point, Integral Type, $\alpha P_{f,g}$–Integral Contractive Mappings, Irrational Type, Rational Type

1. Introduction and Preliminaries

1. In 2002, Branciari\(^1\) introduced the notion of contractive mappings of integral type in complete metric spaces. Afterwards, many researchers extended this result to more general contractive conditions of integral type providing sufficient assumptions which ensure the existence and uniqueness of fixed points (see for example Gupta and Saxena\(^2\), Liu et al.\(^3\), Pathak\(^4\), Rhoades\(^5\) and Vishal and Ashima\(^11\) Vetro).

Throughout this paper, let us consider the following sets:

$$\mathcal{Q}_1 = \left\{ \varphi: \mathbb{R}_+ \to \mathbb{R}_+ \text{ is locally Lebesgue integrable function and } \int_0^\infty \varphi(t) dt > 0 \text{ for each } \varepsilon > 0 \right\},$$

$$\mathcal{Q}_2 = \left\{ \varphi: \mathbb{R}_+ \to \mathbb{R}_+ \text{ is a lower semi-continuous function with } \varphi(0) = 0, \varphi(t) > 0 \text{ for each } t > 0 \text{ and } \lim_{t \to \infty} \varphi(t) > 0 \right\}.$$  

In 2014, Liu and al.\(^6\) extended a result established in the paper\(^7\) by proving the following theorem:

**Theorem 1.1.** Let $(\varphi, \psi)$ be in $\mathcal{Q}_1 \times \mathcal{Q}_2$, $M \in \{M_1, M_2, M_3, M_4\}$ and $T$ be a mapping from a complete metric space $(X, d)$ into itself satisfying

$$\int_0^{d(Tx, Ty)} \varphi(t) dt \leq \int_0^{M(x, y)} \varphi(t) dt + \int_0^{\varphi(M(x, y))} \varphi(t) dt.$$

Then $T$ has a unique fixed point $a \in X$ such that $\lim_{n \to \infty} T^n x = a$ for each $x \in X$.

Where

$$M_1(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} [d(Tx, y) + d(x, Ty)] \right\},$$

$$M_2(x, y) = \max \left\{ d(x, Tx), d(y, Ty) \right\},$$

$$M_3(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty) \right\},$$

$$M_4(x, y) = \max \left\{ d(x, Tx), d(y, Ty), \frac{1}{2} [d(Tx, y) + d(x, Ty)] \right\}.$$  

In\(^8\), Dass and Gupta proved a fixed point theorem by considering a class of mappings where the contractive inequality depends on rational expressions:

$$d(Tx, Ty) \leq M_5(x, y) := ad(y, Ty) + \frac{1}{2} |d(Tx, y) d(x, y)|.$$

A generalization of the above contraction was suggested by Gupta and Saxena\(^9\) (among others, see in particular\(^10\) and\(^11\)) by introducing the following class of mappings:
Another well-known theorem was established by Jaggi\(^1\) which may be considered as an extension of the two last results. He considered the following class of mappings:

\[
d(x,Ty) \leq a_1 d(y,Ty) + a_2 \frac{d(x,Ty)}{d(x,y)} + a_3 d(x,y).
\]

To prove the main results in\(^1\) and\(^\text{12}\), authors impose a very strong and unwanted condition: "T is continuous". These results have been generalized in many ways over the years but the condition of continuity appeared in all the subsequent papers using this type of contractions, as an essential condition.

In our paper, we introduce the concept of $\alpha P_{i_d}$ integral contractive mappings, which is a new class of integral contractive mappings and using this notion we establish a new fixed point theorem. This class of contractions extends and generalizes more or less known results including all previous results\(^\text{2,8,9,12}\). In particular, we omit the condition of continuity of the mapping "T" from the main result in\(^1\) (see corollary 1). Moreover, by corollaries 2, 3, we introduce a new class of rational and irrational contractive mappings and give their related theorems.

Throughout this paper, we consider the following notations:

**Notation 1.2.** Let $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}_+$ be a function. $\Pi_{f(x,y,z,t)}$ denotes the set defined by:

$$
\Pi_{f(x,y,z,t)} := \{k ; k \in \{x,y,z,t\} \text{ and } k \text{ appear in the equation of } f(x,y,z,t)\}.
$$

**Example 1.3.** Let $f : \mathbb{R}^3 \times \mathbb{R}^* \to \mathbb{R}_+$ be a function defined by:

$$
f(x,y,z,t) = x + t
$$

then $\Pi_{f(x,y,z,t)} = \{x,t\}$.

**Notation 1.4.** We denote by $Y$ the set of all functions $f : \mathbb{R}^3 \times \mathbb{R}^* \to \mathbb{R}_+$ satisfying the following assumptions:

\(H_1\)

- Function $(x,y,z) \to f(x,y,z)$ is non-decreasing in each of its variables.

\(H_2\)

- There exists $k \in \{x,y\} ; \{z,k\} \subset \Pi_{f(x,y,z,t)}$.

\(H_3\)

- For all sequences $\{x_n\}, \{x_n^2\}, \{x_n^3\}, \{x_n^4\}$ in $\mathbb{R}_+$, there exists $\delta \in [0,1]$ such that, for large values of $n$:

\[
(H_{2a}) \quad f(x_n) \geq \delta \min \Omega - \left\{0, x_n^4\right\} \text{ if } \mu = 0.
\]

\[
(H_{2b}) \quad f(x_n) \geq \delta \min \left\{\xi, x_n^3; x_n^4 \in \prod f(x_n) \land \xi = \xi_4\right\} \text{ if } f \text{ one of the following conditions hold true:}
\]

- $\mu \neq 0, \xi = \xi_4$ and $\xi_4 = \xi_2 = 0$.
- $\mu \neq 0, \xi_4 \in \{\xi_4, \xi_2\}$ and $x_n^4 \leq 2x_n^4$. \[\text{(H}_{2c}\text{)} \quad f(x_n) \leq \max \Omega \text{ if one of the following conditions hold true:}
\]

- $\{x_n^4 \leq \min \{x_n^4, 2x_n^4\}\}$ and there exists $i \in \{1,3\}$ such that $\xi_i = \mu \neq 0$ and $x_n^4 \in \prod f(x_n)$.
- $\xi_i = \cdots = \xi_4$.

Where

$\Omega = \{1, 2, 3, 4\}$, \quad $f = \max \Omega - \left\{0, x_n^4\right\}$, \quad $x_n^4 \in \prod f(x_n)$.

**Definition 1.5.** Let $f : \mathbb{R}^3 \times \mathbb{R}^* \to \mathbb{R}_+$ be a function. If there exist two functions $g, h$ in $Y$ such that

$$
g(x) \leq f(x) \leq h(x)
$$

for all $x \in \mathbb{R}^3 \times \mathbb{R}^*$, function $f$ is called $Y$-admissible.

**Notation 1.6.** For all $f : \mathbb{R}^3 \times \mathbb{R}^* \to \mathbb{R}_+$, we denote

$$
P_f(x,y) = f(d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(Tx,y)]). \quad (1.1)
$$

**Remark 1.7.** Let us observethat the following mappings $g_i : \mathbb{R}^3 \to \mathbb{R}_+$:

- $g_1(x, y, z, t) = \max \{x, y, z, t\}$, \quad $g_2 = (x, y, z, t) = \max \{y, z\}$, \quad $g_3(x, y, z, t) = \max \{x, z\}$, \quad $g_4(x, y, z, t) = \max \{y, z, t\}$, \quad $g_5(x, y, z, t) = ax + t + y$.

satisfy assumptions $(H_1) - (H_2)$ and for all $i = 1, \ldots, 6$, we have

$$
P_{g_i}(x,y) = g_i(d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(Tx,y)]) = M_i(x, y).
$$
Notation 1.8. We denote by \( \theta : \mathbb{R}_+ \to \mathbb{R}_+ \) a function satisfying the following conditions
\[ A_1 - \theta \text{ is non-decreasing and the sequence } \{\theta_n(t)\}_{n \geq 1} \text{ defined by } \theta_n(t) := \theta^n(t) \text{ (} n \text{th iterate of } \theta) \text{ is bounded for all } t \in \mathbb{R}_+. \]
\[ A_2 - \text{ There exists } (\varepsilon, r) \in [0,1] \times [1,\infty], \text{ for all } (t, \eta) \in \mathbb{R}_+^* \times [0,1] \cup [\varepsilon, r] \cup (\varepsilon, r). \]

Remark 1.9. As we can easily verify, condition \( (A_2) \) imply that
1. For all \( a \in \mathbb{R}_+ \) we have \( \lim_{t \to a} \theta(t) \leq \theta(a) \).
2. \( \theta(0) = 0 \).
3. In particular, \( \theta \) is continuous from the right at the point 0.

Remark 1.10. All \((c)\)-comparison function satisfies conditions \( A_1 \) and \( A_2 \).

Notation 1.11. Let \( T \) be a mapping from a complete metric space \( (X, d) \) into itself, we denote by \( \Gamma(T) \) the set of all functions \( \alpha : X \times X \to \mathbb{R}_+ \) satisfying the following conditions
1. \( \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1, \)
2. \( \alpha(x, y) \geq 1 \]
3. \( \lim_{n \to \infty} \alpha(x_n, y_n) = 0 \Rightarrow \liminf_{n \to \infty} \alpha(x_n, y_n) \geq 1, \)
for all \( x, y, z \in X \) and \( \{x_n\}, \{y_n\} \) nonnegative sequences.

The following lemmas play an important role to obtain our result.

Lemma 1.12. Let \( \phi \in \Phi_1 \) and \( \{r_n\}_n \) be a non-negative sequence with \( \lim_{n \to \infty} r_n = a \). Then
\[ \lim_{n \to \infty} \int_0^{r_n} \phi(t) dt = \int_0^a \phi(t) dt. \]

Lemma 1.13. Let \( \phi \in \Phi_1 \) and \( \{r_n\}_n \) be a non-negative sequence. Then
\[ \lim_{n \to \infty} \int_0^{r_n} \phi(t) dt = 0, \]
if and only if \( \lim_{n \to \infty} r_n = 0 \).

Lemma 1.14. Let \( \phi \in \Phi_1 \) and \( \{u_n\}_n, \{v_n\}_n \) be two nonnegative sequences such that \( u_n \leq v_n \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} u_n = \lim_{n \to \infty} v_n = l \in \mathbb{R}_+ \), then
\[ \lim_{n \to \infty} \int_{u_n}^{v_n} \phi(t) dt = 0. \]

Proof. The condition \( \lim_{n \to \infty} u_n = \lim_{n \to \infty} v_n = l \in \mathbb{R} \) with the fact that \( \phi \in \Phi_1 \) means that
\[ \forall n \in \mathbb{N}, \exists n_0 \geq n, \forall m \geq n_0 ; \int_{u_n}^{v_n} \phi(t) dt \leq \int_{l}^{\frac{1}{n}} \frac{1}{m} \phi(t) dt \]
\[ = \int_{0}^{\frac{1}{n}} \phi(t) dt - \int_{0}^{\frac{1}{m}} \phi(t) dt \]
This, with lemma 1.12, clearly drives to the result.

Lemma 1.15. Let \( \phi \in \Phi_1 \) and \( \{r_n\}_n \) be a nonnegative bounded sequence and \( a, b \in \mathbb{R}_+^* \) such that
\[ \int_0^{r_n} \phi(t) dt \leq \int_0^a \phi(t) dt - a \int_0^b \phi(t) dt, \]
for all \( n \in \mathbb{N} \). Then
\[ \exists N \in \mathbb{N} \mid \forall n \geq N ; r_n = 0. \]

Proof Let \( \phi \in \Phi_1 \) and \( \{r_n\}_n \) be a nonnegative bounded sequence. Assume that there exist \( a, b \in \mathbb{R}_+^* \), such that
\[ \int_0^{r_n} \phi(t) dt \leq \int_0^a \phi(t) dt - a \int_0^b \phi(t) dt. \]

Taking into account condition \( A_1 \) and the fact that \( \{r_n\}_n \) is bounded, we get
\[ \int_0^{r_n} \phi(t) dt \leq \int_0^a \phi(t) dt - a \int_0^b \phi(t) dt. \]

Without loss of generality, we can suppose that
\[ \int_0^{r_n} \phi(t) dt \geq a \int_0^b \phi(t) dt. \]

Then, putting \( \delta_1 := 1 - \left( a \int_0^b \phi(t) dt / \int_0^{\sup \{r_n\}} \phi(t) dt \right) \) and using condition \( A_2 \), we obtain
\[ \int_0^{r_n} \phi(t) dt \leq \delta_1 \left( \int_0^{r_n} \phi(t) dt \right) \]
\[ = \int_0^{\sup \{r_n\}} \phi(t) dt - a \int_0^b \phi(t) dt. \]

Continuing in this way, we obtain
\[ \int_0^{r_n} \phi(t) dt \leq \delta_1 \sum_{n=1}^{N} \delta_1 \left( \int_0^{r_n} \phi(t) dt \right) - a \int_0^b \phi(t) dt. \]

From the fact that the sequence \( \{\theta_n(t)\}_n \) is bounded for all \( t, \theta_1 < 1 \) and \( r \geq 1 \), we deduce that the right hand side of the last inequality tends to \(-a \int_0^b \phi(t) dt \) while \( n \to \infty \), which is a clear contradiction with the fact that \( a, b \in \mathbb{R}_+^* \) and \( \phi \in \Phi_1 \).
We next define the concept of $aP_{\phi}$-integral contractive mappings:

**Definition 1.16.** Let $(\phi, \psi)$ be in $\Phi_1 \times \Phi_2$, $f, g : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+$, $a : X \times X \rightarrow \mathbb{R}_+$ be a functions and $T$ a mapping from a complete metric space $(X, d)$ into itself satisfying the integral inequality:

$$a(x, y) \int_0^t \phi(t) dt \leq a(t) \int_0^t \phi(t) dt - \int_0^t (\phi(t) - \phi(t)) \rho(t) dt, \quad (1.2)$$

for all $x, y$ in $X$. Then $T$ is called $aP_{\phi}$-integral contractive mapping.

### 2. Main Result

**Theorem 2.1.** Let $T$ be an $aP_{\phi_1, \phi_2}$ integral contractive mapping from a complete metric space $(X, d)$ into itself, where $a$ is in $I(T)$ and $f_1, f_2$ are two $\mathcal{Y}$-admissible functions. If $a(x, T(x)) \geq 1$ for some $x \in X$ then $T$ has a fixed point $x \in X$ and the sequence $\{T^n(x_0)\}$ converges to $a$. Moreover, if for all fixed points $x$ and $y$ of the mapping $T$, we have $a(x, y) \geq 1$, the mapping $T$ has a unique fixed point.

**Proof.** First suppose -without loss of generality- that

$$x_n \neq x_{n+1} \text{ and } a(x_n, x_m) \geq 1, \quad (2.1)$$

for any positive integers $n, m$, where $x_{n+1} := T^n x_0$. On the other hand, by the facts that $f_1, f_2$ are two $\mathcal{Y}$-admissible functions, they can be regarded as two elements of $T$ without any loss of generality.

Now, denoting by $d := d(x_n, x_{n+1})$, let’s show that the sequence $\{d\}$ is decreasing to 0 after some rank. For this, assume that for all $n \in \mathbb{N}$, there exists $n > n_0$ such that $d \leq d_{n+1}$. The case $\{d\}$ is an unbounded sequence leads to an obvious contradiction. In fact, consider a non-decreasing subsequence $\{d_{n_k}\}$ with the following properties:

- $d_n$ tends to $+\infty$,
- $d_{n_k} > d_{n_{k-1}}$ for all $k \in \mathbb{N}$.

Invoking assumptions $(H_{1a,1b,2a,2b})$ and relation (1.1) we obtain

$$P_{\phi_2}(x_{n-1}, x_n) = f_2 \left( d_{n-1}, d_{n-1}, d_n, \frac{1}{2} d(x_{n-1}, x_{n+1}) \right) \geq f_2 \left( 0, 0, d_n, \frac{1}{2} (d_n - d_{n-1}) \right) \geq \epsilon d_n,$$

which with the fact that $\psi \in \Phi_2$ means

$$\lim_{k \rightarrow \infty} \psi \left( P_{\phi_2}(x_{n_k}, x_{n_{k+1}}) \right) \in [0, +\infty].$$

Moreover, due to assumptions $(H_{1a,2a})$ and the first property of the sequence $\{d_{n_k}\}$ we have

$$P_{f_i}(x_{n-1}, x_n) \leq f_i \left( d_{n-1}, d_{n-1}, d_n, \frac{1}{2} (d_n + d_{n-1}) \right) \leq f_i(d_n, d_n, d_n, d_n) \leq d_n.$$

The last two results, with the integral inequality (1.2), relation (2.1) and condition $A_i$, imply that

$$\int_0^t \phi(t) dt \leq \int_0^t \phi(t) dt - \int_0^t \phi(t) dt$$

which is in contradiction with Lemma 1.15. Now, returning to identity 1.1 and assumptions $(H_{1a,2c})$ we easily obtain

$$P_{f_i}(x_n, x_{n+1}) \leq \max \{d_n, d_{n+1}\},$$

and by assumptions $(H_{1a,1b,2a,2b})$, we get

$$P_{f_i}(x_n, x_{n+1}) = f_i \left( d_n, d_n, d_n, \frac{1}{2} d(x_n, x_{n+1}) \right) \geq \max \left\{ f_i \left( d_n, d_n, \frac{1}{2} |d_n - d_{n+1}| \right), f_i \left( 0, 0, \frac{1}{2} |d_n - d_{n+1}| \right) \right\} \geq \delta \left( \min \{d_n, d_{n+1}\} \right) \cap \mathbb{R}^+,$$

where $\delta : \liminf_{n \rightarrow +\infty} d_n$. Consequently

$$\delta \left( \min \{d_n, d_{n+1}\} \right) \leq P_{f_i}(x_n, x_{n+1}) \leq \max \{d_n, d_{n+1}\}, \quad (2.2)$$

for all $n \in \mathbb{N}$ and $i = 1, 2$. Doing the recap of the foregoing, we can assert that there exists a sub-sequence $\{d_{n_k}\}$ such that

$$d_{n_k} \leq d_{n_k+1}, \quad d_k \leq M, \quad \lim_{k \rightarrow \infty} d_{n_k} = \left( \liminf_{k \rightarrow \infty} d_{n_k+1} \right),$$

and

$$\frac{\delta \epsilon}{2} \leq \delta \left( \min \{d_{n_k}, \epsilon \} \right) \leq P_{f_i}(x_n, x_{n+1}) \leq d_{n_k+1} \leq M,$$

for sufficiently large values of $k$ and $i = 1, 2$. Using this last result and having in mind relations (1.2), (2.1) and condition $A_i$, we can write
\[
\int_{t_0}^{d_{n+1}} \varphi(t)dt \leq \vartheta \left( \int_{t_0}^{d_{n+1}} \varphi(t)dt \right) - \int_{t_0}^{\varphi(d_{n+1})} \varphi(t)dt.
\]

Remember that \( \partial_{\varphi} c \leq P_{\varphi}(x_{n_{i}}, x_{n_{i+1}}) \leq M \), and in view of the assumed lower semi-continuity of \( \varphi \), we deduce that there exists \( a_{i} \in [\partial_{\varphi} c, M] \) such that 
\[
\varphi(a_{i}) = \inf_{x \in [\partial_{\varphi} c, M]} \varphi(x),
\]
therefore
\[
\int_{t_0}^{d_{n+1}} \varphi(t)dt \leq \vartheta \left( \int_{t_0}^{d_{n+1}} \varphi(t)dt \right) - \int_{t_0}^{\varphi(a_{i})} \varphi(t)dt,
\]
which is a contradiction with lemma (1.15). Thus the sequence \( \{d_{n}\} \) is decreasing and it’s bounded below by 0, hence \( \{d_{n}\} \) converges to some \( c \geq 0 \), we write for sufficiently large values of \( n \)
\[
d_{n} > d_{n+1}, \quad \text{and} \quad \lim_{n \to \infty} d_{n} = c. \tag{2.3}
\]

Taking into account (2.3) and the double inequality (2.2), we obtain for sufficiently large values of \( n \) that
\[
\partial_{\varphi} c \leq P_{\varphi}(x_{n_{i}}, x_{n_{i+1}}) \leq 2c,
\]
then there exists \( a_{i} \in [\partial_{\varphi} c, 2c] \) such that 
\[
\varphi(a_{i}) = \inf_{x \in [\partial_{\varphi} c, 2c]} \varphi(x).
\]
Now, supposing that \( c > 0 \). By virtue of (1.2) and (2.2) it follows that
\[
\int_{t_0}^{d_{n+1}} \varphi(t)dt \leq \Theta \left( \int_{t_0}^{d_{n+1}} \varphi(t)dt \right) - \int_{t_0}^{\varphi(a_{i})} \varphi(t)dt,
\]
from which with relations (2.1), (2.3) and lemma (1.15) we deduce that \( \varphi(a_{i}) = 0 \) and consequently \( c = 0 \), which means
\[
\lim_{n \to \infty} d_{n} = 0. \tag{2.4}
\]

Now, we need to assert that the sequence \( \{x_{n}\}_{n} \) has the Cauchy property. Assume the contrary. Then, by virtue of the last limit, we can extract two subsequences \( \{x_{n_{k}}\}_{k} \) and \( \{x_{m_{k}}\}_{k} \) from the sequence \( \{x_{n}\}_{n} \) such that
\[
\exists \varepsilon > 0, \forall \varepsilon > 0, \exists N_{w} \in \mathbb{N}, \forall k \geq N_{w}, \exists m_{k} \geq k,
\]
\[
d\left( x_{n_{k}}, x_{m_{k}} \right) > \varepsilon, \tag{2.5}
\]
with
\[
d\left( x_{n_{k}-1}, x_{m_{k}} \right) \leq \varepsilon, \tag{2.6}
\]
and
\[
d\left( x_{i}, x_{i+1} \right) < w_{\delta N_{w}}, \forall i \geq N_{w}. \tag{2.7}
\]

With consideration to estimates (2.5) and (2.7) we deduce that
\[
d\left( x_{n_{k}-1}, x_{m_{k}} \right) \geq d\left( x_{n_{k}-1}, x_{m_{k}} \right) - d\left( x_{n_{k}-1}, x_{m_{k}} \right) - d\left( x_{n_{k}-1}, x_{m_{k}} \right) \\
\geq \varepsilon - d\left( x_{n_{k}-1}, x_{m_{k}} \right) - d\left( x_{n_{k}-1}, x_{m_{k}} \right) \\
\geq \varepsilon - 2w_{\delta} \geq \frac{\varepsilon}{4}.
\]

And, by (2.6), that
\[
d\left( x_{n_{k}-1}, x_{m_{k}} \right) \leq d\left( x_{n_{k}-1}, x_{m_{k}} \right) + d_{m_{k}} \leq \varepsilon + w_{\varepsilon},
\]

Which gives
\[
\frac{\varepsilon}{4} \leq d\left( x_{n_{k}-1}, x_{m_{k}} \right) \leq \varepsilon + w_{\varepsilon}, \tag{2.8}
\]
for all \( k \geq N_{w} \). Similarly we obtain
\[
\frac{1}{2} \left[ d\left( x_{n_{k}-1}, x_{m_{k}} \right) + d\left( x_{n_{k}-1}, x_{m_{k}} \right) \right] \leq d\left( x_{n_{k}-1}, x_{m_{k}} \right) + d_{m_{k}} \leq \varepsilon + 2w_{\varepsilon}.
\]

**Remark 2.2.** This inequality implies that
\[
\frac{1}{2} \liminf_{k \to \infty} \left[ d\left( x_{n_{k}-1}, x_{m_{k}} \right) + d\left( x_{n_{k}-1}, x_{m_{k}} \right) \right] \leq \liminf_{k \to \infty} d\left( x_{n_{k}-1}, x_{m_{k}} \right) \leq \varepsilon + 2w_{\varepsilon}.
\]

On the other hand, we have
\[
\frac{1}{2} \left[ d\left( x_{n_{k}-1}, x_{m_{k}} \right) + d\left( x_{n_{k}-1}, x_{m_{k}} \right) \right] \geq d\left( x_{n_{k}-1}, x_{m_{k}} \right) - \frac{d_{m_{k}} + d_{m_{k}}}{2} \geq \frac{\varepsilon}{8}.
\]

Then
\[
\frac{\varepsilon}{8} \leq \frac{1}{2} \left[ d\left( x_{n_{k}-1}, x_{m_{k}} \right) + d\left( x_{n_{k}-1}, x_{m_{k}} \right) \right] \leq \varepsilon + 2w_{\varepsilon}. \tag{2.9}
\]

Now, thanks to relations (1.1), (2.4), (2.8), (2.9), remark 2.2, assumptions \( H_{1,2,3} \) and studying the two cases: \( x \in \Pi_{f_{i}(x,y,z)} \) and \( x \notin \Pi_{f_{i}(x,y,z)} \), it follows that
\[
P_{f_{i}}(x_{n_{k}-1}, x_{m_{k}}) \leq d\left( x_{n_{k}-1}, x_{m_{k}} \right) + d\left( x_{n_{k}-1}, x_{m_{k}} \right) + \frac{d\left( x_{n_{k}-1}, x_{m_{k}} \right)}{2}
\]
\[
\leq \max \left\{ d\left( x_{n_{k}-1}, x_{m_{k}} \right), d\left( x_{n_{k}-1}, x_{m_{k}} \right) + d\left( x_{n_{k}-1}, x_{m_{k}} \right), \right\}
\]
\[
\leq \varepsilon + 2w_{\varepsilon}, \tag{2.10}
\]
for all \( k \geq N_w \). Let’s suppose that \( \{ x, t \} \cap \Pi_{f(x,y,z,t)} = \emptyset \), hence going back to relations (1.2), (2.1), (2.3), condition \( A_1 \), we obtain for \( n < m \)

\[
\int_0^d(x_{n+1},x_{n+1}) \varphi(t) dt \leq \Theta \left( \int_0^d(x_{n+1},x_{n+1}) \varphi(t) dt - \int_0^d(x_{n+1},x_{n+1}) \varphi(t) dt \right)
\]

invoking assumption \( (H_2) \), we derive

\[
\int_0^d(x_{n+1},x_{n+1}) \varphi(t) dt \leq \Theta \left( \int_0^d \varphi(t) dt\right).
\]

Which – in view of relation (2.4) and remark 1.9 - means that the sequence \( \{ x_n \} \), has the Cauchy property. Assume now that \( \{ x, t \} \cap \Pi_{f(x,y,z,t)} \neq \emptyset \). By (2.4), (2.8), (2.9), remark 2.2 and assumption \( (H_3) \), we estimate

\[
p_j \left( x_{n+1},x_{n+1} \right) \geq f \left[ \left( x_{n+1},x_{n+1} \right), \varphi(x) \right] \delta_{n,m} \min \left[ d(x_{n+1},x_{n+1}), d(x_{n+1},x_{n+1}) \right] \leq \delta_{n,m} \min \left[ d(x_{n+1},x_{n+1}), d(x_{n+1},x_{n+1}) \right]
\]

continuing in this way, we obtain

\[
\int_0^d \varphi(t) dt \leq \Theta \left( \int_0^d \varphi(t) dt \right) - \int_0^d \varphi(t) dt.
\]

Having in mind conditions \( A_1 \) and \( A_2 \) we get for sufficiently small values of \( \varepsilon \) and \( w_t \)

\[
\int_0^d \varphi(t) dt \leq \Theta \left( \int_0^d \varphi(t) dt \right) - \int_0^d \varphi(t) dt.
\]

The left hand side of the previous inequality tends to \(-\int_0^d \varphi(t) dt \) while \( m \to \infty \), or

\[
\int_0^d \varphi(t) dt \leq -\int_0^d \varphi(t) dt,
\]

which is a contradiction with the fact that \( \varepsilon > 0 \) and \((\varphi, \psi) \in \Phi_\times \Phi_\times \). This contradiction, with the fact that \( X \) is complete - leads to the result \((\{ x_n \} \) is a convergent sequence). Consider

\[
a := \lim_{n \to \infty} x_n.
\]

From estimates (2.10)–(2.11) and invokes the fact that \( \psi \) is a lower semi-continuous function, we deduce that there exists \( a_2 \in \left[ \delta_{n,m} \frac{\varepsilon}{8}, \varepsilon + 2w_e \right] \) such that,

\[
\varphi(a_2) = \inf_{x \in \left[ \delta_{n,m} \frac{\varepsilon}{8}, \varepsilon + 2w_e \right]} \varphi(x)
\]

this together with relations (1.2), (2.5), (2.10) and condition \( A_1 \) yields

\[
\int_0^d \varphi(t) dt \leq \Theta \left( \int_0^d \varphi(t) dt \right) - \int_0^d \varphi(t) dt
\]

which with condition \( A_2 \) gives

\[
\int_0^d \varphi(t) dt \leq \Theta \left( \int_0^d \varphi(t) dt \right) - \int_0^d \varphi(t) dt,
\]

where

\[
\Theta := 1 + \left( \int_0^{\varepsilon + 2w_e} \varphi(t) dt \right) \rightarrow w_e \to 0 1^*.
\]
\[ \varphi(a) = \inf_{x \in \mathcal{A}} \{d(a, x) \mid d(a, x) = d(x, a)\}. \]

This result with (1.2), (2.13) and condition \( A_1 \) imply that
\[ a(x_n, a) \int_0^{d(x_n, T a)} \varphi(t) \, dt \leq \Theta \left( \int_0^{d(a, T a)} \varphi(t) \, dt \right) - \int_0^{d(a, a)} \varphi(t) \, dt, \]
for sufficiently large values of \( k \). Observing that
\[ d(x_m, Ta) \geq d(a, Ta) - d(x_{m+1}, a) \]

The previous inequality gives
\[ a(x_n, a) \int_0^{d(a, T a)} \varphi(t) \, dt \leq \Theta \left( \int_0^{d(a, T a)} \varphi(t) \, dt \right) - \frac{1}{2} \int_0^{d(a, a)} \varphi(t) \, dt. \]

for sufficiently large values of \( k \). Using again the fact that
\[ \lim_{k \to \infty} x_m = a \] and that \( a \in \Gamma(T) \), we obtain
\[ \int_0^{d(a, T a)} \varphi(t) \, dt \leq \Theta \left( \int_0^{d(a, T a)} \varphi(t) \, dt \right) - \frac{1}{4} \int_0^{d(a, a)} \varphi(t) \, dt. \]

This is a contradiction with lemma 1.15. Then \( a \) is a fixed point of \( T \). To conclude the proof, we need to show that under assumption: “for all fixed points \( x \) and \( y \) of the mapping \( T \), we have \( a(x, y) \geq 1 \)”, \( a \) is the unique fixed point of \( T \). Assume the contrary, i.e.
\[ \exists b \neq a \in X; Ta = a, Tb = b \text{ and } d(a, b) \geq 1. \]

Going back to 1.2 we can write
\[ a(x_n, b) \int_0^{d(x_n, b)} \varphi(t) \, dt = a(x_n, b) \int_0^{d(x_n, b)} \varphi(t) \, dt \]
\[ \leq \Theta \left( \int_0^{P_{a,b}^{[x_n,b]}} \varphi(t) \, dt \right) - \int_0^{P_{a,b}^{[x_n,b]}} \varphi(t) \, dt. \]

Noting first that if \( \{x, t\} \cap \prod f_{(x,y,z)} = \emptyset \) by (1.2) and \((H_{1a,2})\), relation (2.16) become
\[ a(x_n, b) \int_0^{d(x_n, b)} \varphi(t) \, dt \leq \Theta \left( \int_0^{f_{(a,d,a,d)}(x_n, b)} \varphi(t) \, dt \right) - \int_0^{f_{(a,d,a,d)}(x_n, b)} \varphi(t) \, dt \]
\[ \leq \Theta \left( \int_0^{f_{(a,d,a,d)}(x_n, b)} \varphi(t) \, dt \right) - \int_0^{f_{(a,d,a,d)}(x_n, b)} \varphi(t) \, dt \]
\[ \leq \Theta \left( \int_0^{f_{(a,d,a,d)}(x_n, b)} \varphi(t) \, dt \right) - \int_0^{f_{(a,d,a,d)}(x_n, b)} \varphi(t) \, dt \]

Having in mind that \( \liminf_{n \to \infty} a(x_n, b) \geq 1 \) we get
\[ \exists N \in \mathbb{N}, \forall n \geq N; d(x_n, a) \geq 1 \]

From which, with the fact that \( a(x, b) \geq 1 \) we deduce that
\[ a(x, b) \geq 1 \] for sufficiently large values of \( n \). Using this result, lemma 1.12, remark 1.9 and passing to the limit as \( n \) tends to infinity in the last inequality, we get
\[ \int_0^{d(a, b)} \varphi(t) \, dt \leq 0, \]

which is a contradiction with the fact that \( a \neq b \). Then we can suppose that
\[ \{x, t\} \cap \prod f_{(x,y,z)} \neq \emptyset. \]

Relations (1.1), (2.4) together with assumptions \((H_{1a,2})\) allow us to write
\[ P_{f_j}^{(x_n, b)} = f_j \left( d(x_n, b), d_{n-1}^{[a]} \right), \]
\[ \leq \max \left\{ d(x_n, b), d_{n-1}^{[a]} \right\} \]
\[ f_j \left( d_{n-1}^{[a]} \right), \]
\[ \leq \max \left\{ d(x_n, b), d_{n-1}^{[a]} \right\} \]

Where
\[ \zeta = \max \left\{ d(x_n, b), d_{n-1}^{[a]} \right\}. \]

Therefore
\[ P_{f_j}^{(x_n, b)} \leq \max \left\{ d(x_n, b), d_{n-1}^{[a]} \right\} \leq 2d(a, b), \]

for sufficiently large values of \( n \) and \( i = 1, 2 \), which means that the sequence \( \{P_{f_j}^{(x_n, b)}\}_n \) is bounded. On the other hand, thanks to relations (1.1)(2.19) and assumption \((H_{1a})\) we deduce that
\[ P_{f_j}^{(x_n, b)} = f_j \left( d(x_n, b), d_{n-1}^{[a]} \right), \]
\[ \geq \delta \min \left\{ d(a, b), d(x_n, b), d_{n-1}^{[a]} \right\} \]
\[ \geq \delta \min \left\{ d(a, b), d_{n-1}^{[a]} \right\}. \]
from which

\[ P_{L_1}(x_n, b) \geq \frac{1}{2} \delta d(a, b). \]

Then, there exists \( a_2 \in \left[ \frac{1}{2} \delta, d(a, b), 2d(a, b) \right] \) such that \( \varphi(a_2) = \inf_{x \in \left[ \frac{1}{2} \delta, d(a, b), 2d(a, b) \right]} \varphi(x) \). Now, recalling relations (2.15), (2.16), (2.18) and examining the two cases \( d(x_{n+1}, b) \geq d(x_n, b) \) and \( d(x_n, b) \geq d(x_{n+1}, b) \), we get either

\[ \int_0^{d(x_{n+1}, b)} \varphi(t) dt \leq \int_0^{d(x_n, b)} \varphi(t) dt - \int_0^{\varphi(a_2)} \varphi(t) dt \]

Or

\[ \int_0^{d(x_{n+1}, b)} \varphi(t) dt \leq \int_0^{d(x_n, b)} \varphi(t) dt - \int_0^{\varphi(a_2)} \varphi(t) dt, \]

for sufficiently large values of \( n \). From where, we get a contradiction with Lemma 1.3. This finishes the proof.

3. Application

From the main theorem, we can easily get the following corollaries (we omit its proof for simplicity):

**Corollary 3.1.** Let \((\varphi, \psi)\) be in \( \Phi_1 \times \Phi_2 \) and \( T \) be a mapping from a complete metric space \((X, d)\) into itself satisfying the integral inequality

\[ \int_0^{\varphi(Tx, Ty)} \varphi(t) dt \leq \int_0^{N(x, y)} \varphi(t) dt - \int_0^{\varphi(M(x, y))} \varphi(t) dt, \]

for all \( x, y \) in \( X \). Where

\[ N(x, y) = a_1 d(y, Ty) \frac{1 + d(x, Tx)}{1 + d(x, y)} + a_2 \frac{d(x, Ty) + d(y, Ty)}{d(x, y)} + a_3 d(x, y), \]

if \( x \neq y \) and \( N(x, y) = d(x, y) + d(y, Ty) \) if \( x = y \), \((a_1, a_2, a_3) \in \mathbb{R}_+ \times \left( \mathbb{R}_+ \right)^2 \) are constants with \( a_1 + 2a_2 + a_3 < 1 \) and

\[ M \in \{M_1, M_2, M_3, M_4, M_5, M_6\}. \]

Then \( T \) has a unique fixed point \( a \in X \). Moreover, for all \( a \in X \), the sequence \( \{T^n a\} \) converges to \( a \).

**Remark 3.2.** This corollary extend both main results in\(^\text{3,9}\) at the case where mapping \( T \) is not continuous.

**Corollary 3.3.** Let \((\varphi, \psi)\) be in \( \Phi_1 \times \Phi_2 \) and \( T \) be a mapping from a complete metric space \((X, d)\) into itself satisfying the integral inequality

\[ \int_0^{\varphi(Tx, Ty)} \varphi(t) dt \leq \int_0^{N(x, y)} \varphi(t) dt - \int_0^{\varphi(M(x, y))} \varphi(t) dt, \]

for all \( x, y \) in \( X \). Where

\[ N(x, y) = \max \left\{ d(x, y), \frac{d(x, Tx) d(y, Ty)}{d(x, y)}, d(y, Ty) \right\}, \]

if \( x \neq y \) and \( N(x, y) = d(x, y) + d(y, Ty) \) if \( x = y \) and

\[ M \in \{M_1, M_2, M_3, M_4, M_5, M_6\}. \]

Then \( T \) has a unique fixed point \( a \in X \). Moreover, for all \( a \in X \), the sequence \( \{T^n a\} \) converges to \( a \).

**Corollary 3.4.** Let \((\varphi, \psi)\) be in \( \Phi_1 \times \Phi_2 \) and \( T \) be a mapping from a complete metric space \((X, d)\) into itself satisfying the integral inequality

\[ \int_0^{\varphi(Tx, Ty)} \varphi(t) dt \leq \int_0^{N(x, y)} \varphi(t) dt - \int_0^{\varphi(M(x, y))} \varphi(t) dt, \]

for all \( x, y \) in \( X \). Where

\[ N(x, y) = a_1 d(y, Ty) \frac{1 + d(x, Tx)}{1 + d(x, y)} + a_2 \frac{d(x, Ty) + d(y, Ty)}{d(x, y)} + a_3 d(x, y), \]

if \( x \neq y \) and \( N(x, y) = d(x, y) + d(y, Ty) \) if \( x = y \), \((a_1, a_2, a_3) \in \mathbb{R}_+ \times \left( \mathbb{R}_+ \right)^2 \) are constants with \( a_1 + 2a_2 + a_3 < 1 \) and

\[ M \in \{M_1, M_2, M_3, M_4, M_5, M_6\}. \]

Then \( T \) has a unique fixed point \( a \in X \). Moreover, for all \( a \in X \), the sequence \( \{T^n a\} \) converges to \( a \).
for all $x, y$ in $X$. Where

$$N(x, y) = 2a_1d(x, Tx)\frac{d(x, Tx) - d(x, y)}{d(x, Ty) + d(Tx, y)} + a_2d(y, Ty) + a_3d(x, y),$$

if $x \neq y$ and $N(x, y) = d(x, y) + d(y, Ty)$ if $x = y$, $(a_1, a_2, a_3) \in \mathbb{R}_+ \times \mathbb{R}_+^2$ are constants with $a_1 + a_2 + a_3 < 1$ and

$$M \in \{M_1, M_2, M_3, M_4, M_5, M_6\}.$$

Then $T$ has a unique fixed point $a \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}_n$ converges to $a$.

4. References