1. Introduction

The theoretical study of the dynamics of a population leads to complex models due to the inherent complexity of the phenomenon through the interactions and the state dimension. In our study we will present a model describing the dynamics of an epidemic of HIV / AIDS in four municipalities in the city of Tamanrasset (Southern Algeria): this choice is due to several factors:

- Geo strategic position
- Migration flows (mixing national domestic, foreign-national)
- 1200km border between Niger and Mali for the city of Tamanrasset where the specific program for Tamanrasset in the south of Algeria.

2. Epidemiological Model

We will present in this section a mathematical model, considering the different epidemiological and demographic processes and in the end we will introduce an equilibrium model.

2.1 Model Description

Let 'S' means likely classified into three groups rated «S_i».

S_1: ANC (prenatal visit).
S_2: PS (sex workers)
S_3: MSM (men who have sex with men)

And each “i” is made up of “j” subgroups with j = 1, ... 6.
For j = 1: “age” classified k = 1, ...7.
For j = 2: “NI: education” ranked k = 1, ..6.
For j = 3: MS: marital status “ranked k = 1, ..4.
For j = 4: “Sex” classified k = 1, ..., 2.
For j = 5: “Socio-professional class” in classified k = 1, 2.
For j = 6: “nationality” k = 1, ..., 2.
Or k_j number of classes in each subgroup.

The model is presented in the following diagram:

The system of differential equations:

\[ S_{1j} = \Delta_{1j} - \mu_1 S_{1j} - F(S,I)S_{1j} + \Delta S_i, \]
\[ S_{2j} = \Delta_{2j} - \mu_2 S_{2j} - F(S,I)S_{2j}, \]
\[ S_{3j} = \Delta_{3j} - \mu_3 S_{3j} - F(S,I)S_{3j}, \]
\[ I_{1j} = F(S,I)S_{1j} - (\mu_4 + \delta) I_{1j}, \]
\[ I_{2j} = F(S,I)S_{2j} - (\mu_5 + \delta) I_{2j}, \]
\[ I_{3j} = F(S,I)S_{3j} - (\mu_6 + \delta) I_{3j}, \]
\[ R = \delta(I_{1j} + I_{2j} + I_{3j}) - \mu R + \Delta_s, \]

Where \( F(S,I) \) the force of infection defined by:

\[ F(S,I) = \alpha_S S + \alpha_I I + \alpha \]
Modeling of an Epidemiological Model

With $S = \sum_{i=1}^{3} S_i; I = \sum_{j=1}^{3} I_j$ and $\alpha$: Parameter of infection outside.

Note:
All parameters in this system are positive because they represent positive quantities with a biological significance. Let the vector $w = (S_{1,k}, S_{2,k}, S_{3,k}, I_{1,k}, I_{2,k}, I_{3,k}, R)$ under the initial condition $w(0)$ states that variable is the size of the population, it is assumed that
The model structure ensures that the state variables remain non-negative over time.

2.2 Time Scale
In this epidemic, epidemiological parameters are faster than the demographic parameters, which we introduce a lover has time parameter (Scaling parameter) $\varepsilon$ as:

$\alpha_5 = \frac{\alpha_5}{\varepsilon}; \alpha_1 = \frac{\alpha_1}{\varepsilon}; \delta = \frac{\delta}{\varepsilon}$ with $0<\varepsilon<<1$.

Considering a population size, $N_m$ it ensures that the parameters $N_m\alpha_5, N_m\alpha_1, \delta$ are the same order as the parameters of the model.

The parameter $\alpha$ is very small compared to other epidemiological parameters and its order is less than the order of population parameters. Finally, $\delta$ it ensures that the smallest order. In this case the previous system has two time scales. It is reasonable to consider that people spend more time in class than that of likely infected during this time the withdrawal process is spreading faster than infection, which implies that $\delta$ has the greatest order that $N_m\alpha_5, N_m\alpha_1$, and the remaining parameters in the model. In this case we introduce a second parameter of time $\varepsilon'$ as

$\delta = \frac{\delta}{\varepsilon}$ with $0<\varepsilon'<<\varepsilon<<1$.

And in this case we can obtain three time scales.

Note:
All these assumptions about magnetizer parameters will be verified by their numerical values that provide a time scale that will be distinguished by numerical simulations.

2.3 Equilibrium
The epidemiological model has two equilibria; a trivial equilibrium, corresponding to the extinction of the population (free equilibrium) this will be the point stable endemic. Our system has a single endemic, that is because all parameters are positive.

In the system (1), there is no extinction of the population, and is not a zero balance. If we consider $\alpha = 0$ (no infection outside), the equilibrium is given by:

$w' = (S_{1,k}, S_{2,k}, S_{3,k}, 0,0,0,R')$

And one can easily find $w^*$

3. The Order of Scale Model
Using different time scale parameters, we can apply the “theory of singular perturbations” to approximate the original system by another system of lower dimension. Whereas a standard singular perturbation $x = f(t,x,z,\varepsilon); x \in \mathbb{R}^n; x(t_0) = \xi(\varepsilon)$.

$\varepsilon z = g(t,x,z,\varepsilon); z \in \mathbb{R}^m; z(t_0) = \phi(\varepsilon)$

Let $z = h(t,x)$ the quasi-equilibrium solution $g(t, x, z, 0)$

The reduced system is $x = f(t, x, h(t,x), 0)$

Taking $y = zh(t, x)$, we defined the boundary layer system:

$\frac{dy}{du} = g(t, x, y+h(t, x), 0)$

The singular perturbation theory gives the following result (Khalil 1996)

Theorem 1: Approximation for $\varepsilon$ small enough
Suppose that the conditions are met for all $(t,x,z-h(t,x), \varepsilon) \in [0, \infty], B, B, [0, \varepsilon_0, [...$

- The functions $f$, $g$ and their respective partial derivatives $(x, z)$ are continuous and bounded.
- The function $h(t, x)$ and the Jacobean $(\frac{\partial g(t,x,z,\varepsilon)}{\partial z})$ has a partial derivative bounded.
- The Jacobean $\frac{\partial f(t,x,h(t,x),0)}{\partial x}$ is bounded.
- Initial conditions $\xi(\varepsilon)$ and $\phi(\varepsilon)$ and are regular features $\varepsilon$.
- The origin of the reduced system(b) is exponentially stable.
- Origin of the boundary layer system(c) exponentially stable uniformly in$(t,x)$. Then:
∃ positifs constants μ₁, μ₂, and ε’ as
∀∥ξ(0)∥ < μ₁,∥y(0)−h(t₀−ξ(0))∥ < μ₁ and
0< ε << ε’ we obtain:

The singular perturbation in (a) has a unique solution
x(t, ε) z(t, ε) defined ∀ t ≥ t₀ ≥ 0; and x(t,ε)-x’(t)=θ(ε)
z(t,ε)-h(t, x’(t))= y’(t/ε) =0(ε).
Uniformly for all t, ∈[t₀, ∞[, where x’(t) et y’(t) are
solution of (b).
In addition to all t ≥ t₀ et ε ≤ ε’ we have:
z(t,ε)-h(t, x’(t)) = θ(ε) converges uniformly for tt,
∈[t₀, ∞[.

Theorem 2: Stability for ε small enough.

- the equation g(t, x, z, 0)= 0 has an isolated root z=h(t, x)
such that h(t, 0) = 0
- functions f, g and h and their second order partial
derivatives are bounded for
- z-h(t,x) ∈ B₂,
- the origin of (b) is exponentially stable.
- the origin of (c) exponentially stable uniformly in(t, x).
- the origin of the initial system(a) exponentially stable.

3.1 Two Time Scales
Either a change of variable:
A₁ = S₁₁ + I₁₁ + R.
A₂ = S₁₂ + I₁₂.
A₃ = S₁₃ + I₁₃.

This change allows us to identify slow and fast
variables, and each fast variable is expressed in terms of
slow variables, to keep our system and we can minimize it.
The fast variables arrive faster at quasi-equilibrium, so
our system can be written as follows:

A₁ = Δ₁ - μ₁A₁ + δ(t₁j₁₂ + t₁j₁₃) - μR + Δ₁
A₂ = Δ₂ - μ₂A₁ + δ(t₂j₁₂)
A₃ = Δ₃ - μ₃A₁ - δ(t₂j₁₃)

ε₁t₁j₁₂ = εF(S₁₁,t₁,t₃,e₁)(A₁ - R - I₁) - δ' I₁ - μ₁e₁
ε₁t₁j₁₃ = εF(S₁₃,t₁,t₃,e₁)(A₃ - I₁) - δ' I₃ - μ₃e₁

R = Δ + δ'(I₁ + I₂ + I₃) - μR

Where εF(S₁₁,t₁,t₃,e₁) = α₁ + S₁₁ + e₁ + εα =
Assume that X=(A₁, A₂, A₃, R) and Z=(I₁, I₂, I₃).

This system takes the form of an autonomous model
singular perturbation.
X = f(t,x, ε ) ;
εZ = g(x,z,ε) (a).

In this case X and Z are respectively the slow and fast
variables. The system (b) has a single non-negative quasi-
equilibrium when ε →0.

Z* = (I₁₁*, I₁₂*, I₃*) = h(X)=(0,0,0)

Substituting this quasi-equilibrium in the slow
system(a) we obtain the reduced linear system
A₁ = Δ₁ - μ₁A₁ - μR + Δ₁
A₂ = Δ₂ - μ₂A₂
A₃ = Δ₃ - μ₃A₃
R = Δ₃ - μR

Which is a linear system easier to study its stability.
And the equilibrium point is not trivial(0,0,0,0) but
a non-zero point. Under certain conditions there is an equilibrium state non-negative $X^* = (A_1^*, A_2^*, A_3^*, R^*)$ with

$$A_1^* = \frac{\Delta_1}{\mu_1}; A_2^* = \frac{\Delta_2}{\mu_2}; A_3^* = \frac{\Delta_3}{\mu_3}; R^* = \frac{\Delta_R}{\mu}$$

4. Conclusion

- Theorem 1) the model gives a good approximation of the initial system with $Z=Z^* = h(X)$
- Theorem 2) gives an endemic of the initial system and globally-stable.

And after some matrix computations we can easily find the Jacobian of this matrix.

Idea 2

$R_{\text{inconnu}}$: AIDS unknown

$R_{\text{connu}}$: with AIDS known to the system

The system is written as follows:

$$\dot{S}_{1\beta_1} = \Delta S - \mu S + F(S,I)S_{1\beta_1}$$
$$\dot{S}_{2\beta_2} = \Delta S - \mu S + F(S,I)S_{2\beta_2}$$
$$\dot{S}_{3\beta_3} = \Delta S - \mu S + F(S,I)S_{3\beta_3}$$
$$\dot{I}_{1\beta_1} = F(S,I)S_{1\beta_1} - (\mu_1 + \delta)I_{1\beta_1}$$
$$\dot{I}_{2\beta_2} = F(S,I)S_{2\beta_2} - (\mu_2 + \delta)I_{2\beta_2}$$
$$\dot{I}_{3\beta_3} = F(S,I)S_{3\beta_3} - (\mu_3 + \delta)I_{3\beta_3}$$
$$\dot{R}_{\text{conv}} = \delta (I_{1\beta_1} + I_{2\beta_2} + I_{3\beta_3}) - (\mu + \beta)R_{\text{conv}} + \Delta R_{\text{conv}}$$
$$\dot{R}_{\text{inc}} = \beta R_{\text{inc}} + \Delta R_{\text{inc}} - \mu R_{\text{inc}}$$

The slow variables are: $S_{\beta_1}$; $R_{\text{inconnu}}$ and

The fast variables are: $I_{\beta_1}$; $R_{\text{conv}}$

We made the following change of variables:

$$A_1 = S_1 + I_1 + R_{\text{inc}}$$
$$A_2 = S_2 + I_2$$
$$A_3 = S_3 + I_3$$

And write the system as before, replacing the values of $A_1, A_2, A_3, R_{\text{inc}}, \epsilon I_1, \epsilon I_2, \epsilon I_3, \epsilon R$.

When $\epsilon \rightarrow 0$, we found $I_1 = I_2 = I_3 = 0$.

$X^* = (A_1^*, A_2^*, A_3^*, R_{\text{inc}}^*)$ is variable ; and the system can be written:

$$\begin{align*}
A_1 &= \Delta_1 + \Delta_R - \mu R_{\text{inc}} - \mu_1(A_1 - R_{\text{inc}}) \\
A_2 &= \Delta_2 - \mu_2 A_2 \\
A_3 &= \Delta_3 - \mu_3 A_3 \\
R_{\text{inc}} &= \Delta_R - \mu R_{\text{inc}} - \beta R_{\text{inc}}
\end{align*}$$

And we can easily find the Jacobian matrix of the $X^*$.

5. References