Abstract

The article deals with the investigation of the spectral properties of the Dirichlet problem for 2x2 elliptic system of the second order. The need to study the properties of the solvability of boundary value problems for kxk systems of linear differential equations of order n, elliptical, hyperbolic, parabolic, and mixed types appears whenever there is, for example, a problem similar to the one once explored by A.V. Bitsadze. In particular, he pointed to the circumstance in which the Dirichlet problem for elliptic 2x2 system of linear differential equations in partial derivatives was incorrect, which was quite surprising at the time. Important applications of the theory of systems of linear partial differential and the problems associated with the study of the properties of the solvability of boundary value problems formulated to stimulate research relevant spectral problems. Spectral theory of closed differential operators generated by boundary value problems for systems of linear differential equations in partial derivatives started to develop recently. We studied at the same time as the asymptotic behaviour of the own values and the location of the spectrum on the complex plane, and the basic properties of systems composed of vector-functions. Investigation of the structure of the spectrum and the possibility of expanding solutions sets of vector-functions is now one of the main directions in the study of problems of the spectral theory of boundary value problems for systems of linear differential equations in partial derivatives. The carried out research is based on the modified method of model operators.

Keywords: Boundary Value Problems, Closed Operators, Elliptic Systems, Orthogonal Basis, Riesz Basis, Spectrum

1. Introduction

The present work is devoted to a comparative study of the natural and completely describes the spectral properties of a number of closed and at the same time, unrestricted linear differential operators generated studied Dirichlet problem for the 2 × 2 elliptic second-order systems.

The article also deals with the comparative study and description of the spectral properties of differential operators generated by the Dirichlet problem for the elliptic system (1) without the “lower terms”.

\[
\begin{align*}
\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} &= \lambda u_1 + f_1^1, \\
\frac{\partial^2 u_2}{\partial t^2} + \frac{\partial^2 u_1}{\partial x^2} &= \lambda u_2 + f_2^2
\end{align*}
\]  

Moreover, a description of the spectral properties of a differential L operator, generated by the Dirichlet problem (2) - (2*) for the elliptic system (2) with the lower terms of the variables \( t, x \).

\[
\begin{align*}
\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial x} &= \lambda u_1 + f_1^1, \\
\frac{\partial^2 u_2}{\partial t^2} + \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial u_2}{\partial t} + \frac{\partial u_1}{\partial x} &= \lambda u_2 + f_2^2
\end{align*}
\]

To consider the closure \( V_{t,x} \) of restricted area \( \Omega_{t,x} = (0, \pi)^2 \) of Euclidean space \( \mathbb{R}^2_{t,x} \).

If we join to the system of Equations (1) and (2) the Dirichlet condition (2*)

\[
u \big|_{\partial \Omega_{t,x}}
\]  

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We will obtain two boundary problems (1) - (2*) and (2) - (2*).

Thus, two systems of partial differential equations of the second order with Dirichlet conditions with regards to variable \( t \) make the matter for scientific inquiry. The study is aimed at investigating the operator’s spectral properties generated by boundary problems for these systems, namely: the description of the spectrum and the behavior of eigenvector functions.

For the Cauchy-Riemann equations and more general so-called symmetric and asymmetric systems, there are a number of deep results related to the description of the correct boundary conditions. Description of regular boundary value problems for more general systems of equations of the first order on a distinguished variable \( t \) when the number of variables studied by more than two (Kornienko, 2013). Study of the properties of the Dirichlet problem for \((2 \times 2)\) - elliptic systems dedicated to the work (Dezin, 1959). Description of regular boundary work for more general systems of equations of the first order on a distinguished variable \( t \) for those devoted to the work of more than two.

2. Methodology

The modified method of model operators will be called by information technology, which is based on symbolic computation packages, for example, Scailab, Mathematics, Mathlab and others. The use of these packages occurs whenever the study of relevant tasks. The basis of the study, of course, is functional methods that are widely used in their research scholars such as, for example, K. Friedrichs, L. Hermander, S.L. Sobolev, A.A. Dezin, L.P. Tepoyan, V.N. Maslennikova, V.A. Ilyin, V.K. Romanko, E.I. Moiseev, A.P. Soldatov, their disciples and followers.

3. Tensor Product of Hilbert Spaces and Operators

Let \( H' \) u \( H'' \) a pair of separable Hilbert spaces, each of which is given orthonormal basis. Form the Hilbert space \( H \) as follows: as a basis \( H \) let us take the set of ordered pairs \( \varphi_k \otimes \psi_j \), defined for these pairs scalar product according to the rule:

\[
(\varphi_k \otimes \psi_j, \varphi_l \otimes \psi_i) = (\varphi_k, \varphi_l) (\psi_j, \psi_i)
\]

Where on the right are the scalar products in \( H' \) and \( H'' \) respectively. Basis \( \{\varphi_k \otimes \psi_j\}_{k,j} \) - orthonormal. The product is distributed in the usual way on finite linear combinations:

\[
\sum_{i,k} f_{i,k} \varphi_k \otimes \psi_j
\]

Replenishment of the norm introduced a set of finite linear combinations gives a complete Hilbert space \( H = H' \otimes H'' \). Tensor product of Hilbert spaces source.

In accordance with the design for any pair of elements \( f = \sum f_k \varphi_k \in H', g = \sum g_j \psi_j \in H'' \) is determined by their tensor product.

If now \( A': H' \to H'' \) closed linear operator with close to \( H \) domain \( D(A'), \varphi_k \in D(A') \) for any \( k \) and operator \( A': H' \to H' \) has similar properties to that of the dense \( H \) a set of elements (over the set of finite linear combinations) the operator:

\[
A' \otimes A'' \left( \sum_{i,k} f_{i,k} \varphi_k \otimes \psi_j \right) = \sum_{i,k} f_{i,k} A' \varphi_k \otimes A'' \psi_j
\]

Short in \( H \) thus given operator \( A' \otimes A'' \) (With dense domain) is defined by the operator \( A' \otimes A'' H \to H \). If \( H = H' \otimes H'' \) and \( H' \); \( H'' \) - Functional spaces, the \( H' \) can be naturally embedded into \( H \) due to the identification of a subset \( H' \otimes 1 \). Due to the above, the elements \( H' \) are often regarded as elements of \( H \) without any reservations (and without the transition from \( f \) to \( f \otimes 1 \)). The situation is similar with operators \( A': H' \to H' \): They identify with the operators of the form: \( A' \otimes 1 \). The above construction occurs naturally whenever \( H' \); \( H'' \) - our standard Hilbert space of functions over some areas \( V' \subset \mathbb{R}^p \), \( V'' \subset \mathbb{R}^q \). Then \( H \) - adequate space above \( V' \times V'' \). The corresponding operation \( L'(D) \otimes L''(D) \) usually written usually just in the way that will allow the operator to build a model for the \( 2 \times 2 \) elliptical second-order system via a dedicated variable. It is a convenient class of operators, functions of which allow a very simple definition. The difficulties encountered when trying to use the ideal scheme to specific situations arise in the analysis of boundary value problems are related, usually on one side of the complex nature of the corresponding functions \( F(z) \), and the other - the desire to include in the review of the operators, not an \( M \)-operators.

4. Results and Conclusion

Marked with the symbol \( \hat{L} \) operator domain, which
is D, and a set of values determined by the right-hand side of (1), we obtain an elliptic differential operator; this operator is not closed. Applying in \( H_{0}^{1} \), closure standard procedure, we obtain a closed extension \( L \) of the operator \( \hat{L} \). In this case we say that the closed operator \( L: H_{0}^{1} \rightarrow \mathbb{C} \) generated by the task (1) - (2*). We will study its spectrum and the spectral properties of its own vector functions.

Speaking of the spectrum of a closed operator, we follow the terminology used in the monographs. Resolvent set, spectrum, point spectrum, the continuous spectrum, and residual spectrum of \( L \) is denoted by symbols \( p_{L} \), \( \sigma_{L} \), \( \rho_{L} \), \( \sigma_{L} \), \( \rho_{L} \), respectively. The following theorem (1):

**Theorem 1.** Spectrum \( C \sigma_{L} \) of the L operator generated by the operator (1), (3) is closure \( RL \) its point spectrum \( R \sigma_{L} \) point spectrum of the L operator is given by the formula:

\[
\lambda_{m,k} = -k^2 + i(-1)^{n+1} \cdot s^2; \quad m = 1, 2; \quad k \in \mathbb{N}; \quad s \in \mathbb{N} \quad (5)
\]

Own vector function of \( L \) operator belonging to own value (5) can be represented as:

\[
u_{m,k}(t,x) = (i c_{1} + (-1)^{m+1} c_{2}) \sin(kt) \sin(sx)\]

Sequence \( \{\nu_{m,k}(t,x): m = 1, 2; k \in \mathbb{N}; s \in \mathbb{N}\} \) vector-functions of the L operator form an orthogonal basis in the Hilbert space \( H_{0}^{1} \).

### 4.1 Spectral Properties of Boundary Value Problems for Elliptic Systems

The basis for the study of systems of linear differential equations in partial derivatives is a method of model operators designed by A. A. Dezin. This method is widely used V. K. Romanko, A. H. Nazarov, L. P. Teppoyan and other researchers.

The work is devoted to investigation of the spectral characteristics of some boundary value problems for systems of linear differential equations of second order in the selected variable.

Let use the symbols \( c_{i} = (\delta^{i}_{1}, \delta^{i}_{2}) , i = 1, 2 \) for denoting; orthonormal basis of the Euclidean space of column vectors, and through \( U_{k}^{1} \) - Unitary space elements \( u = u^{i} e_{i} + u^{j} e_{j}; \quad u^{i} \in C; k = 1, 2; \) with the scalar product \( \langle u, v \rangle_{U_{k}^{1}} = u^{i} v^{i} + u^{j} v^{j} \). Let \( H_{0}^{1} = L_{2}(V_{k}^{1}) \) - Hilbert space of vector functions \( u: V_{k}^{1} \rightarrow C \), the rate of which is given by:

\[
\|u, H_{0}^{1}\| = \int_{V_{k}^{1}} \|u(\tau, \xi)\|_{U_{k}^{1}}^{2} d\tau d\xi \quad (4)
\]

Also let \( D \) - Linear manifold of smooth complex vector-valued functions belonging to the class \( \Omega_{\infty} \cap C^{2}(\Omega_{\infty}) \) and satisfying the Dirichlet conditions (2*).

To describe the spectral properties of the tasks necessary to describe the space in which they are viewed.

### 4.2 Elliptic System with the Lower Terms of the Variables \( t, x \)

Just as in the case of elliptical without lower member denoted \( L \) operator whose domain is the set \( D \), and the set of values defined by the right part of the system (2), we obtain an elliptic differential operator. This operator is not closed. Applying \( H_{0}^{1} \), closure standard procedure, we obtain an elliptic differential operator; this operator is not closed. Applying \( H_{0}^{1} \), closure standard procedure, we obtain a closed extension \( L \) of the operator \( \hat{L} \). In this case we say that the closed operator, generated by the (2) - (2*). Let us study its spectrum and the spectral properties of its own vector functions. The following theorem (2):

**Theorem 2.** Spectrum \( C \sigma_{L} \) of the L operator generated by the operator (2), (20) includes closure \( RL \) its point spectrum \( R \sigma_{L} \) point spectrum of the L operator is given by the equation (6):

\[
\lambda_{m,k} = i(-1)^{m+1} \left( \frac{1}{4} + s^2 \right) - \left( \frac{1}{4} + k^2 \right) \quad (6)
\]

Own vector function of the L operator belonging to own value (6) can be represented as:

\[
u_{m,k}(t,x) = e^{-\frac{1}{2} t} e^{\frac{i}{2} (c_{1} + i(-1)^{m+1} c_{2}) s} \sin(sx)\]

Sequence \( \{\nu_{m,k}(t,x): m = 1, 2; k \in \mathbb{N}; s \in \mathbb{N}\} \) vector-functions of the L operator form a Riesz basis in the Hilbert space \( H_{0}^{1} \). This basis is not orthogonal.

A study of the spectral properties of boundary value problems for elliptic systems (1) and (2) conducted based on the method of information technology, which is based on the method of model operators designed by A.A. Dezin using an abstract structure of the system (7).

\[
\begin{aligned}
\frac{\partial^{2} u}{\partial t^2} + Bu^2 &= \lambda u^2 + f^1 \\
\frac{\partial^{2} u}{\partial t^2} - Bu^1 &= \lambda u^2 + f^2
\end{aligned} \quad (7)
\]

In a heterogeneous system (7) \( B \) symbol appears or some number or some closed differential operator. Where \( u \) is a vector function: \( u = u(t, x) \quad u = (u^1, u^2)^T \).
Adding to the system of equations (7) Dirichlet conditions (2*), we obtain the boundary value problem (7) - (2*). In the future, for the sake of simplicity, we assume \( \Omega_{r,s} = (0, \pi)^2 \).

The need to study the properties of the solvability of boundary value problems for systems of linear differential equations in partial derivatives arises in the study of various economic, physical, chemical, biological, and social processes and phenomena. Studying the properties of the solvability of boundary value problems for systems of linear differential equations in partial derivatives is quite a challenge. Private and quite important aspect of this study is to describe the task of the spectral properties of the studied systems of linear differential equations in partial derivatives. Mostly it refers to the matrix systems of partial differential equations of order \( m \), hyperbolic, parabolic, elliptic types. It is known, for example, that the Dirichlet problem for elliptic systems of both the first and second order, generally speaking, is not regular. Suitable examples are given in the well-known article (Bitsadze, 1948) AV Bitsadze, “On the uniqueness of solutions of the Dirichlet problem for elliptic partial differential equations” shows that the elliptic system of Equations (8)

\[
\begin{align*}
\frac{\partial^2 u^1}{\partial t^2} - 2\frac{\partial^2 u^1}{\partial t \partial x} - \frac{\partial^2 u^1}{\partial x^2} &= 0 \\
\frac{\partial^2 u^2}{\partial t^2} + 2\frac{\partial^2 u^2}{\partial t \partial x} - \frac{\partial^2 u^2}{\partial x^2} &= 0
\end{align*}
\]

It has infinitely many linearly independent solutions of the form: \( u(t, x) = (u^1(t, x), u^2(t, x))^T \) where

\[
\begin{align*}
u^1(t, x) &= \left( r^{n-1} - \frac{r^{n+1}}{R} \right) \cos(n-1)\varphi, \\
u^2(t, x) &= \left( r^{n-1} - \frac{r^{n+1}}{R} \right) \sin(n-1)\varphi, n \in \mathbb{N}
\end{align*}
\]

Satisfying the Dirichlet condition on the boundary of the circle: \( O = \{ t^2 + x^2 \} < R \). Here, \( r \) and \( \varphi \) are coordinate the polar coordinates of the point \((t, x)\), if the polar axis to take the positive part of the axis \( t \), and for pole - old origin.

The question naturally arises about the “correct” operator of boundary value problems for systems of linear differential equations in partial derivatives. The solution to this problem lies in the way the study of the spectral properties of the model boundary value problems that are widely used in the works by A. A. Dezin (USSR State Prize in 1980 for his monograph “The general theory of boundary issues,” 1980) and V. K. Romanko. In this regard, the author investigated the spectral properties of the boundary value problem under study.

Let us study the spectral properties of differential operators generated by boundary problem (7) - (2*).

Denote \( \epsilon_k = (\delta_{k1}^*, \delta_{k2}^*), k = 1,2 \) orthonormal basis of the Euclidean space \( \mathbb{Z}^2 \) Consisting of the column vectors, and the symbol \( \mathcal{U}^2 \) - Unitary space elements of the form: \( u = u^1 e_1 + u^2 e_2 \), in which the inner product is given by the formula \( (u, v) = u^* v^\top + u^\top v \). Suppose also that, as is common, \( \mathcal{H}_{\mathcal{U}^2}^2 = \mathcal{L}^2(V, \epsilon) \) - Hilbert space of complex vector-valued functions, the rate of which is given by (9)

\[
\left| u, \mathcal{U}_{\mathcal{V}^2} \right|^2 = \int \int_{\mathcal{V}} |u(t, \xi); \mathcal{U}_{\mathcal{V}^2}|^2 d\tau d\xi
\]

In addition, denoted by the symbol \( D \) is a linear manifold of smooth complex-valued vector functions \( u = u(t, x) \) belonging to the class \( C(V, \epsilon) \cap C^2(\Omega_{r,s}) \) and satisfy the conditions (2*). We describe as agreed, the spectral properties of the Dirichlet problem for elliptical system (7). Denote \( \hat{L} \) operator whose domain is \( D \), and the set of values defined by the right part of the system (7), we obtain an elliptic operator; this operator is not closed. Applying in a Hilbert space \( \mathcal{H}_{\mathcal{U}^2}^2 \) closure standard procedure, we obtain a closed extension \( L \), the differential operator \( \hat{L} \).

In this case, we say that the closed operator \( L: \mathcal{H}_{\mathcal{U}^2}^2 \rightarrow \mathcal{H}_{\mathcal{U}^2}^2 \) born from the boundary problem (7) - (2*).

We will study its spectrum and the spectral properties of its own vector functions. Speaking of the spectrum, we adhere to the terminology used in the monographs by 9,10 as well as in the works by 7,11,12,14,15.

Let us turn to a discussion of the main results. The general solution of the homogeneous system (10):

\[
\begin{align*}
\frac{\partial^2 u^1}{\partial t^2} + Bu^1 &= \lambda u^1 \\
\frac{\partial^2 u^2}{\partial t^2} - Bu^1 &= \lambda u^2,
\end{align*}
\]

which belongs to inhomogeneous system (7) is conveniently represented in the form (11):

\[
\begin{align*}
u^1(t) &= -i\left( C_1 e^{\sqrt{-\lambda} t} + C_2 e^{-\sqrt{-\lambda} t} - C_3 e^{\sqrt{-\lambda} t} + C_4 e^{-\sqrt{-\lambda} t} \right) \\
u^2(t) &= \left( C_1 e^{\sqrt{-\lambda} t} + C_2 e^{-\sqrt{-\lambda} t} = C_3 e^{\sqrt{-\lambda} t} + C_4 e^{-\sqrt{-\lambda} t} \right)
\end{align*}
\]
Based on the Dirichlet conditions immediately obtain
\( C_1 + C_2 = C_3 + C_4 = 0 \). Hence, the general solution is
convenient to write as:

\[
\begin{align*}
\mathbf{u}'(t) &= -i\left(C_1 e^{\sqrt{D} t} - C_2 e^{-\sqrt{D} t} - C_3 e^{\sqrt{D} t} + C_4 e^{-\sqrt{D} t}\right), \\
\mathbf{u}'(t) &= \left(C_1 e^{\sqrt{D} t} - C_2 e^{-\sqrt{D} t} + C_3 e^{\sqrt{D} t} + C_4 e^{-\sqrt{D} t}\right).
\end{align*}
\]

As usual, the 4 × 4 solving a system of linear algebraic equations \( u(0) = u(\pi) = 0 \) we write the matrix \( M(\lambda) \). Matrix \( M(\lambda) \) has the form (12):

\[
M(\lambda) = \begin{pmatrix}
-1 & 1 & i & i \\
-i e^{-\sqrt{D} t} & -i e^{\sqrt{D} t} & i e^{-\sqrt{D} t} & i e^{\sqrt{D} t} \\
1 & 1 & 1 & 1 \\
e^{-\sqrt{D} t} & e^{\sqrt{D} t} & e^{-\sqrt{D} t} & e^{\sqrt{D} t}
\end{pmatrix}
\]

We now find the values of the parameter \( \lambda \) in which the determinant of the matrix \( M(\lambda) \) turns to zero. These values are the own values of the differential operator. Determinant \( \Delta(\lambda) \) of matrix \( M(\lambda) \) is convenient to be written in a way that will help to find the own values (13):

\[
\Delta(\lambda) = \left(1 - e^{\sqrt{D} t}\right) \cdot \left(1 - e^{-\sqrt{D} t}\right).
\]

Solving naturally equation (13) regarding \( \lambda \) it is easy to calculate the own values of the differential operator. The following theorem (3):

**Theorem 3.** Spectrum \( \sigma L \) Operator, generated by the (7) (10) consists of a closure \( \overline{RL} \) on the complex plane \( C \) of its point spectrum \( P \sigma L \). Multitude \( C \sigma L = \sigma L \setminus P \sigma L \) forms a continuous spectrum of the L operator. The point spectrum of the L operator is given by the formula:

\[
\lambda_{m,k} = -k^2 + i(-1)^m \cdot s^2. 
\]

**Proof.** It is enough to note that the sequence \( \{u_{m,k}(t,x) : m = 1,2; k \in \mathbb{N}\} \) vector functions

\[
u_{m,k}(t,x) = \left(e_1 - i(-1)^m e_s\right)e^{\frac{1}{2} t}\sin(k t)\sin(s x)
\]

are orthogonal basis in the Hilbert space \( \mathcal{H}_L^2 \).

4.3 Elliptic Systems with Lower Terms

Just as in the case of an elliptic system without a symbol denoting lower terms \( L \) operator whose domain is the set \( D \), and the set of values defined by the right part of the system (17), we obtain an elliptic differential operator; this operator is not closed. Applying \( \mathcal{H}_L^2 \) standard procedure closure obtains an elliptic differential operator; this operator is not closed. Applying \( \mathcal{H}_L^2 \) standard procedure, we obtain a closed extension \( L \) of the operator \( \hat{L} \). In this case we say that the closed operator, generated by the task (17) (20). We will study its spectrum and the spectral properties of its own vector functions. The following theorem (4):

**Theorem 4.** Spectrum \( \sigma L \) of the L operator generated by the operator (17) (3) is closure \( \overline{RL} \) on the complex plane \( C \) of its point spectrum \( P \sigma L \). Multitude \( \sigma L = \sigma L \setminus P \sigma L \) forms a continuous spectrum of the L operator. The point spectrum of the L operator is given by the formula (16):

\[
\lambda_{m,k} = -k^2 + i(-1)^m \left(s^2 - \frac{1}{4}\right).
\]

Vector-function of the operator, owned by own value (16) can be represented in the form (17):

\[
u_{m,k}(t,x) = \left(e_1 - i(-1)^m e_s\right)e^{\frac{1}{2} t}\sin(k t)\sin(s x)
\]

**Proof.** It is enough to note that the sequence \( \{u_{m,k}(t,x) : m = 1,2; k \in \mathbb{N}\} \) vector functions

\[
u_{m,k}(t,x) = \left(e_1 - i(-1)^m e_s\right)e^{\frac{1}{2} t}\sin(k t)\sin(s x)
\]

are orthogonal basis in the Hilbert space \( \mathcal{H}_L^2 \).
We obtain two boundary problems (21), (20) and (19) (20). For the Cauchy-Riemann systems and more common, so-called symmetric and asymmetric systems, there are a number of deep results related to the description of the correct boundary conditions. Description of regular boundary value problems for more general systems of equations of the first order on a distinguished variable \( t \) when the number of variables is dedicated to the work of more than two14. Study of the properties of the Dirichlet problem for the \( 2 \times 2 \) - elliptic systems dedicated to the work (Kornienko, 2013). Description of regular boundary value problems for more general systems of equations is devoted to work 3. However, the spectral properties of the boundary value problems of a different type in the number of variables more than two are almost unknown. Just as in the works12,13 the system of differential equations (21), (19) is called elliptic systems of the first type. Elliptic systems of the second type with the lower terms, in this case the system will be of the form (21), this system is an analogue system

\[
\begin{align*}
\frac{\partial^4 u_1}{\partial t^4} + \frac{\partial^4 u_2}{\partial t^4} - \frac{\partial^4 u_1}{\partial t^2 \partial x^2} + \frac{\partial^4 u_2}{\partial t^2 \partial x^2} + \frac{\partial^4 u_1}{\partial t \partial x^4} + \frac{\partial^4 u_2}{\partial t \partial x^4} &= \lambda u_1 + f^1 \\
\frac{\partial^4 u_2}{\partial t^4} + \frac{\partial^4 u_1}{\partial t^4} - \frac{\partial^4 u_2}{\partial t^2 \partial x^2} + \frac{\partial^4 u_1}{\partial t^2 \partial x^2} + \frac{\partial^4 u_2}{\partial t \partial x^4} + \frac{\partial^4 u_1}{\partial t \partial x^4} &= \lambda u_2 + f^2
\end{align*}
\]  

(21)

Note that the system (19) is equivalent to the system (21) (in the case of \( \lambda = 0 \)) in the sense that after multiplying the first equation (19) - 1, and a formal change \(-f \) Ha \( f \) (due to the arbitrariness of the right), we obtain the system (21). These considerations suggest that the properties of solvability of boundary problems for these systems irrespective of the conditions determining the boundary problem. However, studies in the case of first order elliptic systems show that the spectral properties of the differential operators are different; they are in some sense similar to those differences that emerged when comparing the weakly irregular - strong, as well as in the study of elliptic systems in the works (Aleksieva, 2010, 2013).

Let the symbols \( e_1 = (\delta^i_1, \delta^i_2), k = 1,2 \) orthonormal basis of the Euclidean space \( e^2 \) column vectors, and through \( \mathcal{U}^2 \) - Unitary space elements \( u = u^i e_i + u^e e^e, u^i \in C : k = 1,2 \) with the scalar product \( (u, v; \mathcal{U}^2) = u^i v^i + u^e v^e \). Let \( \mathcal{H}^2 = \mathcal{L}(V_{t,x}) \). Hilbert space of complex vector-valued functions \( u : V_{t,x} \to C^2 \), the rate of which is given by formula (4). Let also \( D \) - linear manifold of smooth complex vector functions \( u = u(t, x) \) belonging to the class \( C(\Omega_{t,x}) \), \( C^2(\Omega_{t,x}) \) of Dirichlet (4).

We describe the first spectral properties of elliptic system of the type without the lower terms.

### 4.4 Elliptic System without the Lower Terms

Denoting by \( \hat{L} \) the operator whose domain is the set \( \mathcal{D} \), and the set of values defined by the right part of the system (18), we obtain an elliptic operator; this operator is not closed. Applying in a Hilbert space \( \mathcal{H}^2 \) closure standard procedure, we obtain a closed extension \( \hat{L} \), the differential operator \( \hat{L} \). In this case, we say that the closed operator \( \hat{L} : \mathcal{H}^2 \to \mathcal{H}^2 \) is born by the boundary problem (18) - (20). We will study its spectrum and the spectral properties of its vector functions.

Let us state in the form of theorems (5) the spectral properties of the Dirichlet problem (18) - (20). The following theorem (5) exists:

**Theorem 5.** Spectrum \( \sigma \) of \( \hat{L} \) of the L operator generated by the operator (18), (20) includes closure \( \hat{L} \) on the complex plane \( C \) of its point spectrum \( \text{P} \). Multitude \( C \sigma \) \( \Lambda = \sigma \L \) \( \text{P} \) \( \sigma \) \( \text{L} \) forms a continuous spectrum of the L operator. The point spectrum of the \( \hat{L} \) operator is given by the formula (22):

\[
\lambda_{m,k,t,x} = k^2 + i(-1)^m \cdot B; \quad B(s) = s^2 m = 1,2; k \in \mathbb{N}; s \in \mathbb{N}
\]

Vector-function of the \( \hat{L} \) operator, owned by own value (22) is given by (23):

\[
u_{m,k,t,x} = (i c_1 + (-1)^m e_1) \sin(kt) \sin(sx)
\]

The sequence of vector-functions of the \( \hat{L} \) operator is an orthonormal basis in the Hilbert space \( \mathcal{H}^2 \).

### 4.5 Dirichlet Problem (19) - (20) for an Elliptic System with Lower Terms

\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}
\]

Just as in the case of an elliptic system without a lower term denoted \( L \) operator whose domain is the set \( \mathcal{D} \), and the set of values defined by the right part of the system (19), we obtain an elliptic differential operator; this operator is not closed. Applying \( \mathcal{H}^2 \) standard procedure closure obtains an elliptic differential operator; this operator is not closed. Applying \( \mathcal{H}^2 \) closure standard procedure, we obtain a closed extension \( L \) of the operator \( \hat{L} \). In this case we say that the closed operator, generated by the (19) - (20). We study the structure of its spectrum and...
the spectral properties of vector-functions. The following theorem (6) exists.

**Theorem 6.** Spectrum $\sigma L$ of the $L$ operator generated by the task (19) - (20) includes closure $\overline{\sigma L}$ on the complex plane $\mathbb{C}$ of its point spectrum $P \sigma L$. Multitude $\sigma L = \sigma L \cup P \sigma L$ forms a continuous spectrum of the $L$ operator. The point spectrum of the $L$ operator is given by the formula (16):

$$\lambda_{m,k} = i(-1)^m \left(\frac{1}{4} + s^2\right) - \left(k^2 + \frac{1}{4}\right), \; m = 1, 2; \; k \in \mathbb{N}; \; s \in \mathbb{N}$$

(24)

Similarly, the spectral properties of a differential operator generated by the Dirichlet problem for systems of the form

5. **References**