Generalization of Rough Topology

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Abstract

Background/Objectives: The essential part of rough set theory is an approximation space. This theory can be defined as lower and upper approximations and these approximations are defined using the equivalence classes. We need to generalize this theory in uncertain environment. Our objective is to propose a generalized rough topology. Methods/Statistical Analysis: Interior and closure of rough topology are same as lower, upper approximations of a rough set. We will find interior and closure of rough topology and obtain extended rough topology. Findings: In this paper, we have studied rough set topologically and proposed a definition of rough fuzzy topology by considering approximations and boundary. Theorems and propositions related to rough fuzzy topology are proposed in this paper. Application/Improvements: Our rough fuzzy topology can be applied to solve real life issues where decision values are in fuzzy form. An algorithm is proposed for the same.

Keywords: Lower Approximations, Rough Fuzzy Topology, Rough Set, Rough Topology, Upper Approximations

1. Introduction

Approximation of crisp set is given in1. Fuzzy set theory, introduced in2, is an extension of classical set theory. Theories of rough sets and fuzzy sets are a distinct and complementary generalization of set theory. The theory of fuzzy set and its new applications are given in3. Rough set theory is a different mathematical tool to deal with vagueness and uncertainty. This theory does not require any beginning or additional information of the data. To express equivalence relation this theory considers indiscernibility relation between objects where as fuzzy set theory does not consider this relation. In fuzzy sets membership values describes uncertainty. It is a mathematical tool for representing uncertain information in reasoning, machine learning, knowledge acquisition, decision analysis, knowledge discovery and decision making.

Rough fuzzy set is a generalized version of a rough set. It is inherited from the approximation in a crisp approximation space. It was introduced in4. The concept of topology exists in almost all branches. In a rough topological space, lower and upper approximations are the same as interior and closure5. Rough set theory is a tool to deal with incomplete and imprecise data and topology is the study of invariance of a space under topological transformations known as homeomorphisms. Some problems related to topological rough space and its topological properties are given in6. Rough set theory is defined in terms of topology in various ways. A new topology introduced in7 is called the rough topology in terms of rough set and is used to analyze the deciding factor for the most common diseases like Chikungunya and Diabetes. Rough topology and basic properties of rough topological space are studied in8. Rough topology and basic properties of rough topological space, such as rough open sets, rough closed sets, rough base and rough closure are studied in9.

Covering-based rough set theory is an extension of classical rough set. It is discussed with topological point of view in10. In11 fuzzy rough set models and fuzzy topologies on finite universe are discussed. We can find relation of fuzzy rough sets, fuzzy closure spaces and fuzzy topology was investigated in12. The basic structure of rough set the-

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ory is an approximation space. Fuzzy form of rough sets and their topological structures are studied in\textsuperscript{13}. Fuzzy rough approximations are further investigated. Rough functions was generalized topologically in\textsuperscript{14}.

A novel location routing protocol that uses smart antennas to estimate nodes positions into the network and to deliver information basing routing decisions on neighbor’s status connection and relative position, named LBRA\textsuperscript{15}. The main purpose of LBRA is to eliminate network control overhead as much as possible. By taking the above aspects, the present analysis is carried out. With the help of above approach rough set is useful tool in medical applications. The adjustable approach for ranking objects based on fuzzy soft models is given in\textsuperscript{16}. Rough set rule induction algorithms are accomplished of generating decision rules which can potentially provide new medical insight and profound medical knowledge\textsuperscript{17}.

In this paper, we have generalized the rough topology considering membership function. Some basic propositions and theorems are proved. We have applied the concept of rough fuzzy topology to solve real life example.

### 2. Preliminaries

In section 2, we recall some basic concepts.

In\textsuperscript{1,18-20} considered universal set to introduce equivalence relation. He called the pair approximation space and defined lower, upper approximations, boundary.

Basic operations on fuzzy sets like union intersection, complement, subsets, equality can be obtained from in\textsuperscript{21}. Fuzzy set becomes crisp set when $\alpha$-cut is applied\textsuperscript{3}.

A generalized version of rough set and definition of rough fuzzy set are given in\textsuperscript{4,22}. It was derived from the approximation of a fuzzy set. The approximation was defined with membership value between zero and one.

The definition of fuzzy topology can be obtained in\textsuperscript{20,23}. He defined interior as join of open sets and meet as closure of closed sets. In topological operators are used to define rough set theory. Author expressed approximation as interior and closure. Rough topology is define in\textsuperscript{7}.

### 3. Rough Fuzzy Topology

In section 3 we have proposed rough fuzzy topology. In addition lower approximations, upper approximations and boundary are defined in fuzzy form.

**Definition 3.1** Rough fuzzy topology:

Consider $U$ as universal set with $R$ an equivalence relation on it. Let $X$ be a fuzzy set on $U$ and $\tau_{RF} = \left\{ 0, 1, (RF)_X, (\overline{RF})_X, BN_{RF}(X) \right\}$, where $(RF)_X$, $(\overline{RF})_X$ and $BN_{RF}(X)$ are lower approximation, upper approximation and boundary respectively. $\tau_{RF}$ satisfies the following three conditions:

(i) $0 \in \tau_{RF}$ and $1 \in \tau_{RF}$,

(ii) If $\mu_i \in \tau_{RF}$ and $\nu_j \in \tau_{RF}$, then $\mu_i \land \nu_j \in \tau_{RF}$,

(iii) If $\mu_i \in \tau_{RF}$ $\forall i \in I$, then $\bigvee I \mu_i \in \tau_{RF}$.

$\tau_{RF}$ is rough fuzzy topology. We call $(X, \tau_{RF})$ as rough fuzzy topological space.

Since

\[
(RRF)_X \subseteq (RRF)_X \Rightarrow \mu_{(RRF)_X}(x) \leq \mu_{(RRF)_X}(x)
\]

\[
(RRF)_X \cup (RRF)_X = \left\{ x, \max\left(\mu_{(RRF)_X}(x), \mu_{(RRF)_X}(x)\right) \right\} = \left\{ x, \mu_{(RRF)_X}(x) \right\} = (RRF)_X \in \tau_{RF},
\]

\[
(RRF)_X \cup BN_{RF}(X) = \left\{ x, \max\left(\mu_{(RF)_X}(x), \mu_{BN_{RF}}(X)(x)\right) \right\} = \left\{ x, \mu_{(RF)_X}(x) \right\} = (RF)_X \in \tau_{RF},
\]

\[
(RRF)_X \cap (RRF)_X = \left\{ x, \min\left(\mu_{(RF)_X}(x), \mu_{(RF)_X}(x)\right) \right\} = \left\{ x, \mu_{(RF)_X}(x) \right\} = (RF)_X \in \tau_{RF},
\]

\[
(RRF)_X \cap BN_{RF}(X) = \left\{ x, \min\left(\mu_{(RF)_X}(x), \mu_{BN_{RF}}(X)(x)\right) \right\} = \left\{ x, \mu_{(RF)_X}(x) \right\} = (RF)_X \in \tau_{RF},
\]
Remark 1

If we apply \( \alpha \)–cut on rough fuzzy topology the lower approximations, upper approximations and boundary reduces to crisp set. Topology \( \tau_{\text{RF}} \) reduces to \( \tau \) as:

\[
\alpha \cap BN_{\text{RF}}X = \{x, \min(\mu_{\text{RF}}(x), \mu_{BN_{\text{RF}}}X(x))\} = \{x, \mu_{BN_{\text{RF}}}X(x)\} = BN_{\text{RF}}(X) \in \tau_{\text{RF}},
\]

\[
\alpha \cup \overline{1} = \{x, \max(\mu_{\text{RF}}(x), \mu_{\overline{1}}(x))\} = \{x, \mu_{\overline{1}}(x)\} = \overline{1} \in \tau_{\text{RF}},
\]

and \( \alpha \cap BN_{\text{RF}}(X) = \{x, \min(\mu_{\text{RF}}(x), \mu_{BN_{\text{RF}}}X(x))\} = \{x, \mu_{\text{RF}}(x)\} = \overline{0} \in \tau_{\text{RF}}. \]

The pair \( (X, \tau_{\text{RF}}) \) is called the rough fuzzy topological space. The elements of \( \tau_{\text{RF}} \) are called fuzzy open sets. A fuzzy set \( K \) is called fuzzy closed if \( cK \in \tau_{\text{RF}} \). We denote by \( \mathcal{F}\tau_{\text{RF}} \) the collection of all fuzzy closed sets in rough fuzzy topological space.

Definition 3.2: Let \( K \) be any fuzzy set then \( K \) is a fuzzy closed if \( cK \in \mathcal{F}\tau_{\text{RF}} \). \( \mathcal{F}\tau_{\text{RF}} \) is collection of all fuzzy closed set in rough fuzzy topological space. We have

(i) \( \overline{0} \in \tau_{\text{RF}} \) and \( \overline{1} \in \tau_{\text{RF}} \),

(ii) If \( \mu, \nu \in \tau_{\text{RF}} \), then \( \mu \land \nu \in \tau_{\text{RF}} \),

(iii) If \( \mu_i \in \tau_{\text{RF}} \), then \( \bigwedge_i \mu_i \in \tau_{\text{RF}} \).

The interior and closure in rough fuzzy topology defined as:

Definition 3.3 Interior and Closure: Let \( X \) be a fuzzy set then

(i) interior of \( X \) is

\[
(\text{RF})X = \{x, \mu_{(\text{RF})X}(x)\}; x \in X \text{ and } \mu_{(\text{RF})X}(x) = \inf_{y \in X} \{\max(y)\},
\]

(ii) closure of \( X \) is

\[
(\overline{\text{RF}})X = \{x, \mu_{(\overline{\text{RF}})X}(x)\}; x \in X \text{ and } \mu_{(\overline{\text{RF}})X}(x) = \sup_{y \in X} \{\min(y)\}.
\]

Definition 3.4: Let \( U \) be a universal set and \( \mathcal{R} \) be an equivalence relation on \( U \). Let \( \tau_{\text{RF}} \) be the rough fuzzy topology on \( U \) and \( M \) be any subset of \( A \). The set of attributes is called the core of \( \mathcal{R} \) if

Base of \( M \neq \text{Base of } \mathcal{R}(r) \forall r \in M. \)

We can find with the help of rough fuzzy topology lower and upper approximations are same as the interior and closure.

Example 1:

Let \( U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \)
\( U/\mathcal{R} = \{\{x_1, x_2\}, \{x_3, x_4, x_5, x_6\}, \{x_7, x_8\}\} \)
\( X = \{(x_1,0.2),(x_2,0.8),(x_3,0.7),(x_4,0.4),(x_5,0.6),(x_6,0.1),(x_7,1),(x_8,0.2)\} \)

Using definition of rough fuzzy set we get:

\[
(\overline{\text{RF}})X = \{(x_1,0.2),(x_2,0.8),(x_3,0.7),(x_4,0.4),(x_5,0.6),(x_6,0.1),(x_7,1),(x_8,0.2)\}
\]

\[
BN_{\text{RF}}(X) = (\overline{\text{RF}})X \ominus (\overline{\text{RF}})X = \{(x_1,0.7),(x_3,0.7),(x_4,0.7)\}
\]

\[
\tau_{\text{RF}} = \{\overline{0}, \overline{1}, (\overline{\text{RF}})X, (\overline{\text{RF}})X, BN_{\text{RF}}(X)\};
\]

\[
(\overline{\text{RF}})X = \{(x_1,0.8),(x_2,0.8),(x_3,0.7),(x_4,0.7),(x_7,0.7),(x_8,0.7),(x_8,1),(x_8,0.2)\}
\]

(9)

Thus, \( \tau_{\text{RF}} \) is a rough fuzzy topology. From Remark 3.1 if \( \alpha = 0.1 \) is applied on \( \tau_{\text{RF}} \) then rough fuzzy topology reduces to \( \tau \), i.e.,

\[
(\text{RF})_\alpha = \{x, x_1, x_2\}, \ (\overline{\text{RF}})_\alpha = \{x, x_1, x_2, x_3, x_4, x_5, x_6\}, BN_{\alpha}(X) = \{x, x_1, x_2\}
\]

\[
(1)_\alpha = U
\]

\[
\tau = \{(1)_\alpha, (\text{RF})_\alpha, (\overline{\text{RF}})_\alpha, BN_{\alpha}(X)\} \text{ is rough topology on } X.
\]

We can find interior and closure of \( \tau_{\text{RF}} \) by membership values. For
\( X = \{(x_1,0.2),(x_2,0.8),(x_3,0.7),(x_4,0.4),(x_5,0.6),(x_6,0.1),(x_7,1),(x_8,0.2)\} \)

Using the definition of interior the membership value of every \( x \in X \) is obtained as:
\[ \mu_{(\text{RF})}(x) = \inf_{y \in X} \max \{ \mu_X(y) \}. \]

Membership value of \( x_i \) and \( x_2 \) are equal as both elements belong in same equivalence class.

\[
\begin{align*}
\mu_{(\text{RF})}(x_i) &= \inf_{y \in X} \max \{ \mu_X(y) \}, \\
\mu_{(\text{RF})}(x_2) &= \inf_{y \in X} \max \{ \mu_X(y) \}, \\
m_{(\text{RF})}(x_i) &= \inf_{y \in X} \max \{ \mu_X(y) \}, \\
m_{(\text{RF})}(x_2) &= \inf_{y \in X} \max \{ \mu_X(y) \}, \\
\end{align*}
\]

Hence, membership value of every \( x \in X \) is given as:

\[
\begin{align*}
\mu_{(\text{RF})}(x) &= 0.2, \\
\mu_{(\text{RF})}(x) &= 0.2, \\
\mu_{(\text{RF})}(x) &= 0, \\
\mu_{(\text{RF})}(x) &= 0.2, \\
\mu_{(\text{RF})}(x) &= 0.2. \\
\end{align*}
\]

From (9) & (10), we find lower approximations of \( \tau_{\text{RF}} \) to be equal to interior of \( \tau_{\text{RF}} \). Hence, lower approximation is equal to the interior in rough fuzzy topology.

The closed sets of topology are defined as:

\[
\bar{\hat{0}}, \bar{\hat{1}}, (\text{RF})X', (\text{RF})X', (\text{BN}_R(X))', \text{ where } \bar{\hat{0}} = 1, \bar{\hat{1}} = 0,
\]

\[
((\text{RF})X) = \{(x_0, 0.8), (x_0, 0.8), (x_0, 0.8), (x_0, 0.8), \}
\]

\[
(((\text{RF})X)' = \{(x_0, 0.8), (x_0, 0.8), (x_0, 0.8), (x_0, 0.8), \}
\]

\[
((\text{BN}_R(X))' = \{(x_4, 0.3), (x_5, 0.3), (x_6, 0.3)\}.
\]

For \( X = \{x_0, x_2\}, \) and by definition of closure we obtain the membership value for every \( x \in X \).

\[
\mu_{(\text{RF})}(x) = \sup_{y \in X} \min \{ \mu_X(y) \} \text{ where } x \in X.
\]

Membership value of \( x_i \) and \( x_2 \) are equal because both elements belong in same equivalence class.

\[
\begin{align*}
\mu_{(\text{RF})}(x_i) &= \mu_{(\text{RF})}(x_i), \\
\mu_{(\text{RF})}(x_i) &= \mu_{(\text{RF})}(x_i), \\
\mu_{(\text{RF})}(x_i) &= \mu_{(\text{RF})}(x_i), \\
\mu_{(\text{RF})}(x_i) &= \mu_{(\text{RF})}(x_i).
\end{align*}
\]

Hence, membership value of every \( x \in X \) is given as:

\[
\begin{align*}
\mu_{(\text{RF})}(x) &= 0.2, \\
\mu_{(\text{RF})}(x) &= 0.2, \\
\mu_{(\text{RF})}(x) &= 0, \\
\mu_{(\text{RF})}(x) &= 0.2, \\
\mu_{(\text{RF})}(x) &= 0.2. \\
\end{align*}
\]

From (10) and (12) we find closure & upper approximations of \( \tau_{\text{RF}} \) to be same.

**Remark 2:**

Let \( \tau_1 \) and \( \tau_2 \) be rough fuzzy topology on \( X \). If \( \tau_1 \subset \tau_2 \) then, we say that \( \tau_2 \) is finer than \( \tau_1 \) and \( \tau_1 \) is coarser than \( \tau_2 \).

Using the definition of rough fuzzy topology we have proposed the following theorem as:

**Theorem 1:** Consider a rough fuzzy topological space \((X, \tau_{RF})\) where, \( X \) is non empty fuzzy finite set and \( \tau_{RF} \) is a rough fuzzy topology on \( X \). Let \( T \) be the collection of all rough fuzzy open subset of \((X, \tau_{RF})\).

Then \( T \) is a fuzzy topology on \( X \).

In a rough fuzzy topology if \( \tau_{RF} \) satisfies the conditions given in theorem 2 then \( \tau_{RF} \) will be a rough fuzzy topology.

**Theorem 2:** Let \( X \) be a non empty finite set and \( \tau_{RF} \) be a family of rough fuzzy closed subset of \((X, \tau_{RF})\). Then \( \tau_{RF} \) satisfies the following properties:

(i) \( \bar{0} \in \tau_{RF} \) and \( \bar{1} \in \tau_{RF} \),

(ii) If \( \mu, \nu \in \tau_{RF} \), then \( \mu \lor \nu \in \tau_{RF} \),

(iii) If \( \mu_i \in \tau_{RF} \lor i \in I \), then \( \mu_i \in \tau_{RF} \).

Proof: Consider \((X, \tau_{RF})\) be a rough fuzzy topological space, where \( X \) be a fuzzy set and \( \tau_{RF} \) be a topology on \( X \) and let \( \tau_{RF} \) be the collection of all fuzzy closed subsets of \((X, \tau_{RF})\).
Consider \( (U, \tau) \) be fuzzy sets on \( U \), then following conditions are satisfied,
\[
\begin{align*}
& (i) \quad 0 \in \tau_{\mathcal{R}} \quad \Rightarrow \quad 0 \in \tau_{\mathcal{R}} \quad \Rightarrow \quad 1 \in \tau_{\mathcal{R}}, \\
& (ii) \quad \mu, \nu \in \tau_{\mathcal{R}} \quad \Rightarrow \quad \mu' \nu' \in \tau_{\mathcal{R}} \quad \Rightarrow \quad \mu \nu \in \tau_{\mathcal{R}}. \\
& (iii) \quad \mu \in \tau_{\mathcal{R}} \quad \forall \ i \in I \quad \Rightarrow \quad \mu_i \in \tau_{\mathcal{R}} \quad \forall \ i \in I. \\
& (iv) \quad (\mathcal{R}(X \cup Y))' \supseteq (\mathcal{R}(X))' \cup (\mathcal{R}(Y))'.
\end{align*}
\]

Proposition 1: Consider \( (U, \mathcal{R}) \) to be an approximation space and \( X \) and \( Y \) are fuzzy sets on \( U \), then following conditions are satisfied,
\[
\begin{align*}
& (i) \quad (\mathcal{R}(X))' \subseteq X \subseteq (\mathcal{R}(X))', \\
& (ii) \quad (\mathcal{R}(\phi))' = (\mathcal{R} \phi) = \phi \quad \text{and} \quad (\mathcal{R}(F)) U = (\mathcal{R}(F)) U = U, \\
& (iii) \quad (\mathcal{R}(X \cup Y))' \supseteq (\mathcal{R}(X))' \cup (\mathcal{R}(Y))', \\
& (iv) \quad (\mathcal{R}(X \cup Y))' = (\mathcal{R}(X))' \cup (\mathcal{R}(Y))'.
\end{align*}
\]

Proof: The proof (i), (ii), (vii), (viii), (ix), (x) are trivial. We have given proof of (iii), (iv), (v), (vi).

(iii) Since \( X \subseteq X \cup Y \) and \( Y \subseteq X \cup Y \)
\[
\begin{align*}
\mu_{\mathcal{R}(X \cup Y)}(x) &= \inf_{x \in X} \max_{x \notin X} \mu_{\mathcal{R}(X \cup Y)}(x) \\
& \geq \inf_{x \in X} \max_{x \notin X} \mu_{\mathcal{R}(X)}(x) \\
& \geq \mu_{\mathcal{R}(X)}(x) \quad (: X \subseteq X \cup Y)
\end{align*}
\]

Similarly, \( \mu_{\mathcal{R}(X \cup Y)}(x) \geq \mu_{\mathcal{R}(Y)}(x) \).

Therefore, \( (\mathcal{R}(X)' \cup \mathcal{R}(Y)) \subseteq (\mathcal{R}(X \cup Y))' \) and \( (\mathcal{R}(X)' \cup \mathcal{R}(Y)) \subseteq (\mathcal{R}(X \cup Y))' \).

Hence, \( (\mathcal{R}(X)' \cup \mathcal{R}(Y)) \subseteq (\mathcal{R}(X \cup Y))' \).

We have illustrated the property with the help of examples.

Let \( U \{x_1, x_2, x_3, x_4, x_5\} \) and equivalence classes of \( \mathcal{R} \) be \( U/\mathcal{R} = \{\{x_1, x_2, x_3\}, \{x_4, x_5\}\} \).

Let \( X = \{x_{(0,0.3)}, x_{(0.6,0.3)}, x_{(0.3,0.1)}, x_{(0.1,0.7)}\} \) and \( Y = \{x_{(0.2,0.3)}, x_{(0.4,0.3)}, x_{(0.1,0.7)}\} \).

\[
\begin{align*}
\mu_{\mathcal{R}(X \cup Y)}(x_1) &= \mu_{\mathcal{R}(X \cup Y)}(x_2) = \mu_{\mathcal{R}(X \cup Y)}(x_3) = 0.6, \\
\mu_{\mathcal{R}(X \cup Y)}(x_4) &= \mu_{\mathcal{R}(X \cup Y)}(x_5) = 0.2, \\
\mu_{\mathcal{R}(X \cup Y)}(x_1) &= \mu_{\mathcal{R}(X \cup Y)}(x_2) = \mu_{\mathcal{R}(X \cup Y)}(x_3) = 0.6, \\
\mu_{\mathcal{R}(X \cup Y)}(x_4) &= \mu_{\mathcal{R}(X \cup Y)}(x_5) = 0.2.
\end{align*}
\]

Similarly, we can find membership value of elements of \( X \) and \( Y \) as
\[
\begin{align*}
\mu_{\mathcal{R}(X \cup Y)}(x_1) &= \mu_{\mathcal{R}(X \cup Y)}(x_2) = 0.1, \\
\mu_{\mathcal{R}(X \cup Y)}(x_3) = 0.5, \\
\mu_{\mathcal{R}(X \cup Y)}(x_1) &= \mu_{\mathcal{R}(X \cup Y)}(x_2) = 0.1, \\
\mu_{\mathcal{R}(X \cup Y)}(x_3) = 0.5.
\end{align*}
\]

Hence, \( (\mathcal{R}(X \cup Y))' \subseteq (\mathcal{R}(X))' \cup (\mathcal{R}(Y))' \).

condition is satisfied.
(iv) \( \mu_{(\mathcal{RF}(X \cap Y))}(x) = \sup_{\alpha \in \mathbb{X}} \min \left\{ \mu_{(\mathcal{RF})x}(x) \right\} \)

\( = \sup_{\alpha \in \mathbb{X}} \left\{ \min \left\{ \max \left( \mu_{(\mathcal{RF})x}(x), \mu_{(\mathcal{RF})y}(x) \right) \right\} \right\} \)

\( = \sup_{\alpha \in \mathbb{X}} \left\{ \max \left\{ \min \left( \mu_{(\mathcal{RF})x}(x), \mu_{(\mathcal{RF})y}(x) \right) \right\} \right\} \)

\( = \sup_{\alpha \in \mathbb{X}} \left\{ \min \left\{ \sup \left( \mu_{(\mathcal{RF})x}(x), \mu_{(\mathcal{RF})y}(x) \right) \right\} \right\} \)

\( = \sup_{\alpha \in \mathbb{X}} \left\{ \min \left( \mu_{(\mathcal{RF})x}(x), \mu_{(\mathcal{RF})y}(x) \right) \right\} \)

\( = \mu_{(\mathcal{RF})x}(x) \cap \mu_{(\mathcal{RF})y}(x) \)

Hence, \((\mathcal{RF}(X \cap Y)) = (\mathcal{RF})X \cap (\mathcal{RF})Y \cap (\mathcal{RF})Y^c \).

(vi) Since \( X \supseteq X \cap Y \) and \( Y \supseteq X \cap Y \)

\( \mu_{(\mathcal{RF}(X \cap Y))}(x) = \inf_{\alpha \in \mathbb{X}} \max \left\{ \mu_{(\mathcal{RF})x}(x) \right\} \)

\( \leq \inf_{\alpha \in \mathbb{X}} \max \left\{ \mu_{(\mathcal{RF})x}(x) \right\} \)

\( \leq \mu_{(\mathcal{RF})x}(x) \)

\( \mu_{(\mathcal{RF}(X \cap Y))}(x) \leq \mu_{(\mathcal{RF})x}(x) \).

Similarly, \( \mu_{(\mathcal{RF}(X \cap Y))}(x) \leq \mu_{(\mathcal{RF})x}(x) \).

Therefore, \((\mathcal{RF})(X \cap Y) \leq (\mathcal{RF})(X \cap Y) \) and \((\mathcal{RF})(Y) \leq (\mathcal{RF})(X \cap Y) \).

Hence, \((\mathcal{RF})(X) \cap (\mathcal{RF})(Y) \leq (\mathcal{RF})(X \cap Y) \).

With the help of example we illustrate the property.

Let \( U = \{x_1, x_2, x_3, x_4, x_5\} \) and equivalence classes of \( \mathcal{R} \) be

\[ U/\mathcal{R} = \{\{x_1, x_2, x_3\}, \{x_4, x_5\}\}. \]

Let \( X = \{x_1, x_2, x_3, x_4\}, Y = \{x_1, x_2, x_3\} \).

\[ \mu_{(\mathcal{RF}(X \cap Y))}(x_1) = \mu_{(\mathcal{RF}(X \cap Y))}(x_2) = \mu_{(\mathcal{RF}(X \cap Y))}(x_3) \]

\[ \mu_{(\mathcal{RF}(X \cap Y))}(x_4) \]

\( = \sup \{\min \{\mu_{X}(y), \mu_{Y}(y)\}\} \)

\( = \{\min \{0.1, 0.2\}, \min \{0.3, 0.3\}, \min \{0.1, 0.2\}\} \)

\( = \{0.1, 0.2, 0.3, 0.3\} \)

\( = 0.3 \).

\( \mu_{(\mathcal{RF}(X \cap Y))}(x_5) \)

\( = \{\min \{0.6, 0.5\}, \min \{0.7, 0.7\}\} \)

\( = 0.7 \).

Similarly we can find the membership values of elements of \( X \) and \( Y \) as

\[ \mu_{(\mathcal{RF})x}(x_1) = \mu_{(\mathcal{RF})x}(x_2) = \mu_{(\mathcal{RF})x}(x_3) = 0.3; \]

\[ \mu_{(\mathcal{RF})x}(x_4) = 0.7. \]

\[ \mu_{(\mathcal{RF})y}(y_1) = \mu_{(\mathcal{RF})y}(y_2) = \mu_{(\mathcal{RF})y}(y_3) = 0.3; \]

\[ \mu_{(\mathcal{RF})y}(y_4) = 0.7. \]

Hence, \((\mathcal{RF})(X \cap Y) \leq (\mathcal{RF})X \cap (\mathcal{RF})Y \cap (\mathcal{RF})Y^c \) condition is satisfied.

**Proposition 2:** If \( \tau_{\mathcal{RF}} \) is the rough fuzzy topology on \( U \) with respect to \( X \), then the set \( \beta = \{\bar{1}, (\mathcal{RF})X, BN_{\mathcal{RF}}(X)\} \) is the basis for \( \tau_{\mathcal{RF}} \).

Proof: (i) \( \cup_{K \in \beta} K = \{x, \max(\mu_k(x)) : K \in \beta\} = \bar{1} \)

(ii) Consider \( \bar{1} \) and \( BN_{\mathcal{RF}}(X) \in \beta \)

Let \( Z = BN_{\mathcal{RF}}(X), Z \subseteq \bar{1} \cap BN_{\mathcal{RF}}(X) \) [:: \( \bar{1} \cap BN_{\mathcal{RF}}(X) = BN_{\mathcal{RF}}(X) \)]
\[(x, \mu(x)) \in \tilde{1} \cap BN_{\text{sf}}(X)\]
\[= (x, \mu(x)) \in \tilde{1} \text{ and (} x, \mu(x) \text{)} \in BN_{\text{sf}}(X)\]
\[= (x, \mu(x)) \in BN_{\text{sf}}(X) \quad [\because \tilde{1} \cap BN_{\text{sf}}(X) = BN_{\text{sf}}(X)]\]

Consider \(\tilde{1}\) and \((RF)X \in \beta\)

Let \(W = (RF)X, \ W \subset \tilde{1} \cap (RF)X\)
\[\therefore \tilde{1} \cap (RF)X = (RF)X\]

\[(x, \mu(x)) \in \tilde{1} \cap (RF)X\]
\[= (x, \mu(x)) \in \tilde{1} \text{ and (} x, \mu(x) \text{)} \in (RF)X\]
\[= (x, \mu(x)) \in (RF)X \quad [\because \tilde{1} \cap (RF)X = (RF)X]\]

Consider \((RF)X \in \beta\),
\[(RF)X \cap BN_{\text{sf}}(X) = \phi.\]

Thus \(\beta\) is a base for \(\tau\).

Example 2:
We can check the important attributes required for decision using rough fuzzy topology. Table 1 represents the medical survey of hospital for flu.

Table 1. Decision table for flu

<table>
<thead>
<tr>
<th>Patients</th>
<th>Headache</th>
<th>Temp.</th>
<th>Flu</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>Low</td>
<td>Normal</td>
<td>0.4</td>
</tr>
<tr>
<td>P2</td>
<td>High</td>
<td>High</td>
<td>0.5</td>
</tr>
<tr>
<td>P3</td>
<td>High</td>
<td>Very-high</td>
<td>0.6</td>
</tr>
<tr>
<td>P4</td>
<td>Low</td>
<td>Normal</td>
<td>0.3</td>
</tr>
<tr>
<td>P5</td>
<td>High</td>
<td>High</td>
<td>0.4</td>
</tr>
<tr>
<td>P6</td>
<td>High</td>
<td>Very-high</td>
<td>0.6</td>
</tr>
<tr>
<td>P7</td>
<td>Low</td>
<td>Very-high</td>
<td>0.5</td>
</tr>
<tr>
<td>P8</td>
<td>High</td>
<td>Very-high</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Columns of table 1 are attributes (Headache, Temperature) and row are objects (Patients). Attributes divides in two parts conditional and decision attributes.

In attribute headache value sets represents by low and high and in attribute temperature values sets represented by normal, high and very-high. In flu (decision attribute) our values are in fuzzy form but in general our decision values are in crisp form. The attribute headache and temperature generate equivalence classes. The data set is inconsistent because the patient 1, 4 and 2, 5 have different membership value therefore the problem cannot be solved exactly it can be solved approximately. By using rough fuzzy topology we will check the attributes headache or temperatures are important for flu or not. We remove one of the attribute in conditional attributes and find its topology and basis, if this topology and basis are not same as the original topology then attributes belongs in core. The membership values for each patient are defined as:

\[U = \{ P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8 \}\]
\[X = \{ (P_1,0.4), (P_2,0.5), (P_3,0.6), (P_4,0.3), (P_5,0.4), (P_6,0.6), (P_7,0.5), (P_8,0.7) \}\]
\[\mathcal{R} = \{ \text{Headache, Temp.} \} = \{ P_1, P_4 \}, \{ P_2, P_3 \}, \{ P_3, P_6, P_8 \}, \{ P_7 \}\]

\[X_1\] belongs only those elements which have the membership value greater than 0.5.

Let \(X_1 = \{ U \mid \text{Flu} (U) = \text{yes} \}\) = \{ P_2, P_3, P_6, P_7, P_8 \}

If \(\alpha = 0.5\) then

\[(RF)X_1 = \{ (P_1,0.3), (P_2,0.4), (P_3,0.6), (P_4,0.3), (P_5,0.4), (P_6,0.6), (P_7,0.5), (P_8,0.6) \}\]

\[(RF)X = \{ (P_1,0.4), (P_2,0.4), (P_3,0.7), (P_4,0.3), (P_5,0.5), (P_6,0.7), (P_7,0.5), (P_8,0.5) \}\]

This topology is the rough topology on \(U\). If we remove the temperature then the corresponding equivalence classes are:

\[\mathcal{R} = \{ \text{Headache} \} = \{ \{ P_1, P_4 \}, \{ P_2, P_3, P_6, P_7, P_8 \} \}\]

\[(RF)X_1 = \{ (P_1,0.3), (P_2,0.4), (P_3,0.4), (P_4,0.3), (P_5,0.4), (P_6,0.4), (P_7,0.3), (P_8,0.4) \}\]

\[(RF)X = \{ (P_1,0.5), (P_2,0.7), (P_3,0.7), (P_4,0.7), (P_5,0.7), (P_6,0.7), (P_7,0.5), (P_8,0.7) \}\]
If $\alpha = 0.5$ then
\[
(\mathcal{R}X_1)_\alpha = \{ \phi \}, \quad (\mathcal{R}X_1)'_\alpha = \{ P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8 \},
\]
\[
BN'_\alpha \left( X_1 \right) = \{ P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8 \}
\]
\[
\tau_{(R-H)} = \{ \phi, U, \{ P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8 \} \}
\]
\[
\beta_{(R-H)} = \{ U, \{ P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8 \} \}
\]
\[
\tau_{(R-H)} \neq \tau_{\mathcal{R}}, \quad \beta_{(R-H)} \neq \beta_{\mathcal{R}}
\]

The above topology and base are not same original topology. So we cannot remove the temperature.

If we remove the headache and applying same analysis then topology and base are not same original topology. So we cannot remove the headache.

Similarly we can apply the same analysis for patients not suffering from flu. From above method, we find that headache and temperature are important factor for flu. This method is applicable if our decision values are fuzzy form.

**Example 3:**

In this example, we consider the ten students with grade of students in two subjects. Using the grades, we convert our decision values in fuzzy form. We can find attributes required for decision using rough fuzzy topology. This table represents the survey of college:

<table>
<thead>
<tr>
<th>Students</th>
<th>Mathematics</th>
<th>Electronics</th>
<th>CGPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$ (11102154)</td>
<td>B$^+$</td>
<td>B$^+$</td>
<td>0.7</td>
</tr>
<tr>
<td>$S_2$ (11102158)</td>
<td>A$^+$</td>
<td>A</td>
<td>0.8</td>
</tr>
<tr>
<td>$S_3$ (11102159)</td>
<td>C$^+$</td>
<td>B$^+$</td>
<td>0.5</td>
</tr>
<tr>
<td>$S_4$ (11102160)</td>
<td>A$^+$</td>
<td>A</td>
<td>0.7</td>
</tr>
<tr>
<td>$S_5$ (11102161)</td>
<td>C</td>
<td>A</td>
<td>0.4</td>
</tr>
<tr>
<td>$S_6$ (11102162)</td>
<td>C$^+$</td>
<td>B$^+$</td>
<td>0.5</td>
</tr>
<tr>
<td>$S_7$ (11102163)</td>
<td>C$^+$</td>
<td>B$^+$</td>
<td>0.4</td>
</tr>
<tr>
<td>$S_8$ (11102164)</td>
<td>B$^+$</td>
<td>B$^+$</td>
<td>0.6</td>
</tr>
<tr>
<td>$S_9$ (11102165)</td>
<td>C$^+$</td>
<td>A</td>
<td>0.5</td>
</tr>
<tr>
<td>$S_{10}$ (11102166)</td>
<td>C$^+$</td>
<td>B$^+$</td>
<td>0.4</td>
</tr>
</tbody>
</table>

In table 2 the grades of subject Mathematics and Electronics are attributes and students are objects. Attributes divides in two parts conditional and decision attributes. In conditional attributes, values are represented by grades. In CGPA (decision attribute) our values are in fuzzy form but in general, our decision values are in crisp form. The attributes Mathematics and Electronics generate equivalence classes. From rough topology, we will check that the attributes mathematics or electronics are important for CGPA or not. The membership values for each student are defined as:

\[
U = \{ S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10} \}
\]

\[
X = \{ (S_1, 0.7), (S_1, 0.8), (S_2, 0.5), (S_3, 0.7), (S_4, 0.4), (S_5, 0.5), (S_6, 0.4), (S_7, 0.6), (S_8, 0.5), (S_9, 0.4) \}
\]

\[
\mathcal{R} = \{ \text{Mathematics, Electronics} \} = \{ \{ S_1, S_2, S_4 \}, \{ S_3, S_5, S_6 \}, \{ S_7, S_8, S_{10} \} \}
\]

$X_i$ belongs only those elements which have the membership value garter than 0.5. They have the good CGPA.

Let $X_1 = \{ U \mid \text{CGPA (U) = good} \} = \{ S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10} \}$

\[
(\mathcal{R}F)X = \{ (S_1, 0.6), (S_2, 0.7), (S_3, 0.4), (S_4, 0.7), (S_5, 0.4), (S_6, 0.4), (S_7, 0.4), (S_8, 0.6), (S_9, 0.5), (S_{10}, 0.4) \}
\]

\[
(\mathcal{R}F)X = \{ (S_1, 0.7), (S_2, 0.8), (S_3, 0.5), (S_4, 0.8), (S_5, 0.4), (S_6, 0.5), (S_7, 0.5), (S_8, 0.7), (S_9, 0.5), (S_{10}, 0.4) \}
\]

If $\alpha = 0.5$ then

\[
(\mathcal{R}X_1)'_\alpha = \{ S_1, S_2, S_4, S_8, S_9 \}, \quad (\mathcal{R}X_1)_\alpha = \{ S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10} \},
\]

\[
BN'_\alpha \left( X_1 \right) = \{ S_1, S_2, S_4, S_8, S_9 \}
\]

\[
\tau_{\mathcal{R}} = \{ \phi, U, \{ S_1, S_2, S_4, S_8, S_9 \}, \{ S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10} \} \}
\]

\[
\beta_{\mathcal{R}} = \{ U, \{ S_1, S_2, S_4, S_8, S_9 \}, \{ S_3, S_5, S_6 \} \}
\]

This topology is the rough topology on U. If we remove the Electronics then the corresponding equivalence classes are:

\[
\mathcal{R} = \{ \text{Mathematics} \} = \{ \{ S_1, S_2, S_4 \}, \{ S_3, S_4, S_5 \}, \{ S_6, S_7, S_8, S_9, S_{10} \} \}
\]
Given a finite universal set $U$, $\mathcal{R}$ an equivalence relation on $U$ and $\chi$ be a fuzzy set. Let $A$ be the set of attributes that is divided into two classes named as $C$ for conditional attributes and $D$ for decision attributes. In table, columns are indicated by attributes, rows by objects. All contents of table are attribute values.

Step 1: Find lower approximation, upper approximations and boundary of the set with the help of $\alpha$-cut (using rough fuzzy set).

Step 2: Find the rough topology $\tau_\alpha$ on $U$ and its basis $\beta_\alpha$.

Step 3: Remove an attribute $x$ from $C$ and find lower approximation, upper approximation and boundary of $x$ with respect to $\mathcal{R}$ on $C - \{x\}$.

Step 4: Find rough topology $\tau_{\mathcal{R}(x)}$ on $U$ and find its basis $\beta_{\mathcal{R}(x)}$.

Step 5: Repeat Step 3 and 4 for all conditional attributes $C$.

Step 6: The attributes in $C$, for which $\text{Base of } \mathcal{R}(x) = \text{Base of } \mathcal{R}$ form the core $\text{(R)}$.

With the help of algorithm we can find important attributes.

5. Conclusions

In this paper, definition of rough fuzzy topology using lower approximation, upper approximation and boundary are proposed. We have proposed related proposition and some theorems of rough fuzzy topology. In real life situations, if our decision values are in fuzzy form then we can solve the problem with the help of rough fuzzy topology. Rough fuzzy topologies are applied in real life problems and algorithm has been proposed for the same. In this situation, we can check the attributes that affect decision.

6. References

7. Thivagar ML, Richard C, Paul NR. Mathematical Innovations of a Modern Topology in Medical Events.