Some Properties of a New Class of Univalent Functions Involving a New Generalized Differential Operator with Negative Coefficients

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Abstract

The primary motivation behind this paper is to present new generalized differential operator 

\[ A^m_{\mu,\lambda,\delta} (a,\beta) f(z) \]

defined through \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) we present a new contribution is subclass of analytic functions \( G_m (\gamma, \sigma, \lambda) \). Additionally, we talk about some properties for univalent functions with many results for the subclass of analytic functions \( G_m (\gamma, \sigma, \lambda) \). Furthermore, with given solve technical to application involving fractional calculus for univalent functions.

Keywords: Analytic Functions, Differential Operator, Close-to-convex Functions, Fractional Calculus, Starlike Functions

1. Introduction and Preliminaries

Now, we will go to present preface for univalent functions, with important contributions to get some important properties in this article.

Let \( A \) denote the class of univalent function \( f(z) \):

\[ f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0) \quad (1) \]

which are analytic function in the unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). For a univalent function \( f \) in \( A \), we introduce the following differential operator

\[ A^0_{\mu,\lambda,\delta} (a,\beta) f(z) = f(z), \]

\[ A^1_{\mu,\lambda,\delta} (a,\beta) f(z) = \left( 1 - \frac{\beta (\lambda - a)}{\mu + \lambda} \right) f(z), \]

\[ + \left( \frac{\beta (\lambda - a)}{\mu + \lambda} \right) z f'(z) + \frac{\delta}{\mu + \lambda} z^2 f''(z), \]

and for \( m = 1, 2, 3, \ldots \)

\[ A^m_{\mu,\lambda,\delta} (a,\beta) f(z) = A^m_{\mu,\lambda,\delta} (a,\beta) \left( A^{m-1}_{\mu,\lambda,\delta} (a,\beta) f(z) \right). \quad (2) \]

If \( f \) is given by (1), then from (2) we get

\[ A^m_{\mu,\lambda,\delta} (a,\beta) f(z) = z + \sum_{n=2}^{\infty} \left[ 1 + \left( \frac{(n-1)(\lambda - a) + n\delta}{\mu + \lambda} \right) \right]^m a_n z^n \quad (3) \]

for \( f \in A, \ a, \lambda, \mu > 0, \beta, \lambda, \mu > 0, \ a \neq \lambda \) and \( m \in \mathbb{N}_0 \).

This generalizes numerous operators as follows.

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(i) $A_{\mu, \lambda, 0}^m (0, 1) f(z) = z + \sum_{n=2}^{\infty} \left[ \frac{\mu + \lambda n}{\mu + \lambda} \right] a_n z^n$, the operator presented and studied by Swamy.

(ii) $A_{\mu, \lambda, 0}^m (a, \beta) f(z) = z + \sum_{n=2}^{\infty} \left[ 1 + (n-1)(\lambda - a) \beta \right] a_n z^n$
the operator presented and studied by Darus and Ibrahim.

(iii) $A_{\mu, \lambda, 0}^m (0, 1) f(z) = z + \sum_{n=2}^{\infty} \left[ 1 + (n-1) \lambda \right] a_n z^n$
the operator presented and studied by Al-Oboudi.

(iv) $A_{0,1,0}^m (0, 1) f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, the operator presented and studied by Salagean.

Let $G_m(\gamma, \sigma, \chi)$ denote and present a new contribution is subclass of $A$ comprising of univalent functions $f$ which satisfy

$$\gamma \left[ A_{\mu, \lambda, \beta}^m (a, \beta) f(z) \right] - \frac{A_{\mu, \lambda, \beta}^m (a, \beta) f(z) z}{\sigma (A_{\mu, \lambda, \beta}^m (a, \beta) f(z) ) (1 - \gamma) A_{\mu, \lambda, \beta}^m (a, \beta) f(z) z} < \chi,$$

where $A_{\mu, \lambda, \beta}^m (a, \beta) f(z)$ is given by (3), $0 \leq \gamma < 1$, $0 \leq \delta < 1$, $0 < \chi < 1$ and for all $z \in U$.

We know that $G_0(\gamma, \sigma, \chi)$ was presented by Atshan and Ghandi. We use techniques similar to these used Amourah et al., Darus and Faisal, Al-Hawary et al., and 13-15.

2. Some Results for the Class $G_m(\gamma, \sigma, \chi)$

In this part in article, we find the first important result coefficientine quality

**Theorem 2.1** A univalent function $f \in A$ is in the class $G_m(\gamma, \sigma, \chi)$ if and only if

$$\sum_{n=2}^{\infty} \left[ \gamma (n-1) + \chi (n \sigma + 1 - \gamma) \right] \left[ 1 + (n-1) \frac{d + \lambda}{\mu + \lambda} \right] a_n \leq \chi (\sigma + 1 - \gamma).$$

The result (5) is sharp.

**Proof.** Suppose that the result (5) hold true and $|z| = 1$.

Now, we follow technique steps

$$\gamma \left[ A_{\mu, \lambda, \beta}^m (a, \beta) f(z) \right] - \frac{A_{\mu, \lambda, \beta}^m (a, \beta) f(z) z}{\sigma (A_{\mu, \lambda, \beta}^m (a, \beta) f(z) ) (1 - \gamma) A_{\mu, \lambda, \beta}^m (a, \beta) f(z) z} < \chi,$$

where $A_{\mu, \lambda, \beta}^m (a, \beta) f(z)$ is given by (3), $0 \leq \gamma < 1$, $0 \leq \delta < 1$, $0 < \chi < 1$ and for all $z \in U$.

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The result (5) is sharp.
Thus

\[
\sum_{n=2}^{\infty} \left[ \gamma (n-1) + \chi (n\sigma + 1 - \gamma) \right] a_n \leq \chi \left( \sigma + (1 - \gamma) \right)
\]

This implies that \( f \in G_m \left( \gamma, \sigma, \chi \right) \). The result for coefficient inequality (5) is sharp for the univalent function

\[
f(z) = z - \chi \left( \sigma + (1 - \gamma) \right)
\]

\[
\left[ \gamma (n-1) + \chi (n\sigma + 1 - \gamma) \right] \left[ 1 + \frac{(n-1) \left( (\lambda - a) \beta + n\delta \right)}{\mu + \lambda} \right] a_n \leq \chi \left( \sigma + (1 - \gamma) \right)
\]

and \( z^n \) (\( n \geq 2 \)).

**Theorem 2.2** If the function \( f \) defined by (1) is in the class \( G_m \left( \gamma, \sigma, \chi \right) \), then for \( |z| < 1 \), we have

\[
|z| + \frac{\chi \left( \sigma + (1 - \gamma) \right)}{\left[ \gamma + \chi \left( 2\sigma + 1 - \gamma \right) \right] \left[ 1 + \frac{(\lambda - a) \beta + 2\delta}{\mu + \lambda} \right]} |z|^2 \leq |f(z)| \leq |z| + \frac{\chi \left( \sigma + (1 - \gamma) \right)}{\left[ \gamma + \chi \left( 2\sigma + 1 - \gamma \right) \right] \left[ 1 + \frac{(\lambda - a) \beta + 2\delta}{\mu + \lambda} \right]} |z|^2
\]

(6)

and

\[
1 - \frac{2\chi \left( \sigma + (1 - \gamma) \right)}{\left[ \gamma + \chi \left( 2\sigma + 1 - \gamma \right) \right] \left[ 1 + \frac{(\lambda - a) \beta + 2\delta}{\mu + \lambda} \right]} |z| \leq |f(z)| \leq 1 + \frac{\chi \left( \sigma + (1 - \gamma) \right)}{\left[ \gamma + \chi \left( 2\sigma + 1 - \gamma \right) \right] \left[ 1 + \frac{(\lambda - a) \beta + 2\delta}{\mu + \lambda} \right]} |z|
\]

(7)

**Proof.** It is easy to see that, for \( G_m \left( \gamma, \sigma, \chi \right) \),

\[
\sum_{n=2}^{\infty} a_n \leq \frac{\chi \left( \sigma + (1 - \gamma) \right)}{\left[ \gamma (n-1) + \chi (n\sigma + 1 - \gamma) \right] \left[ 1 + \frac{(n-1) \left( (\lambda - a) \beta + n\delta \right)}{\mu + \lambda} \right]^m}
\]

And

\[
\sum_{n=2}^{\infty} aa_n \leq \frac{2\chi \left( \sigma + (1 - \gamma) \right)}{\left[ \gamma (n-1) + \chi (n\sigma + 1 - \gamma) \right] \left[ 1 + \frac{(n-1) \left( (\lambda - a) \beta + n\delta \right)}{\mu + \lambda} \right]^m},
\]

we have,

\[
|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n
\]

\[
|z| + \frac{\chi \left( \sigma + (1 - \gamma) \right)}{\left[ \gamma + \chi \left( 2\sigma + 1 - \gamma \right) \right] \left[ 1 + \frac{(\lambda - a) \beta + 2\delta}{\mu + \lambda} \right]} |z|^2
\]

and

\[
|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geq |z| - \frac{\chi \left( \sigma + (1 - \gamma) \right)}{\left[ \gamma + \chi \left( 2\sigma + 1 - \gamma \right) \right] \left[ 1 + \frac{(\lambda - a) \beta + 2\delta}{\mu + \lambda} \right]} |z|^2
\]
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\[ f'(z) \leq 1 + \sum_{n=2}^{\infty} na_n \geq |z| \]

\[ f'(z) \geq 1 - |z| \sum_{n=2}^{\infty} na_n \geq |z| \]

\[ \frac{2\gamma (\sigma + (1-\gamma))}{\gamma + \gamma (2\sigma + 1-\gamma)} \left[ 1 + \frac{(\lambda - a)\beta + 2\delta}{\mu + \lambda} \right]^m |z| \]

\[ \frac{2\gamma (\sigma + (1-\gamma))}{\gamma + \gamma (2\sigma + 1-\gamma)} \left[ 1 + \frac{(\lambda - a)\beta + 2\delta}{\mu + \lambda} \right]^m |z| \]

3. Close-to-convexity, Star likeness and Convexity

In this part in article for a univalent function \( f(z) \), we find the second important result for \( f(z) \in A \) is said to be close-to-convex of order \( \eta \) if it satisfies

\[ \text{Re} \left\{ f'(z) \right\} > \eta \quad (8) \]

for some \( \eta \geq 0 \) and for all \( z \in U \). Also a univalent function \( f(z) \in A \) is said to be starlike of order \( \eta \) if it satisfies

\[ \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \eta \quad (9) \]

for some \( \eta \geq 0 \) and for all \( z \in U \). Additional, a univalent function \( f(z) \in A \) is said to be convex of order \( \eta \), if and only if \( zf'(z) \) is starlike of order \( \eta \), that is

\[ \text{Re} \left\{ 1 + \frac{zf'(z)}{f(z)} \right\} > \eta \quad (10) \]

for some \( \eta \geq 0 \) and for all \( z \in U \).

**Theorem 3.1** If \( f(z) \in G_m (\gamma, \sigma, \chi) \), then a univalent function \( f(z) \) is close-to-convex of order \( \eta \) in \( |z| < h_1(\sigma, \chi, \gamma, \eta) \in G_m (\sigma, \chi, \gamma, \eta) \) where

\[ h_1(\sigma, \chi, \gamma, \eta) = \inf_n \left[ \frac{(1-\eta)\gamma (n-1) + \chi (n\sigma + 1-\gamma)}{1 + \frac{(n-1)(\lambda - a)\beta + n\delta}{\mu + \lambda}} \right]^{1/\eta} \]

**Proof.** It is follow technique steps adequate to demonstrate that

\[ |f'(z) - 1| < \sum_{n=2}^{\infty} na_n |z|^n \leq 1 - \eta \quad (11) \]

And

\[ \sum_{n=2}^{\infty} \left[ \gamma (n-1) + \chi (n\sigma + 1-\gamma) \right] \left[ 1 + \frac{(n-1)(\lambda - a)\beta + n\delta}{\mu + \lambda} \right]^m a_n \leq \chi (\sigma + (1-\gamma)) \]

observe that (11) is true if

\[ \frac{n-1}{\chi (\sigma + (1-\gamma))} \left[ \gamma (n-1) + \chi (n\sigma + 1-\gamma) \right] \left[ 1 + \frac{(n-1)(\lambda - a)\beta + n\delta}{\mu + \lambda} \right]^m a_n \leq \chi (\sigma + (1-\gamma)) \]

Solving (12) for \( |z| \), we obtain

\[ |z| \leq \left[ \frac{(1-\eta)\gamma (n-1) + \chi (n\sigma + 1-\gamma)}{1 + \frac{(n-1)(\lambda - a)\beta + n\delta}{\mu + \lambda}} \right]^{1/\eta} \]

(\( n = 2, 3, \ldots \)).

**Theorem 3.2** If \( f(z) \in G_m (\gamma, \sigma, \chi) \), then a univalent function \( f(z) \) is starlike of order \( \eta \) in \( |z| < h_2(\sigma, \chi, \gamma, \eta) \in G_m (\sigma, \chi, \gamma, \eta) \), where

\[ h_2(\sigma, \chi, \gamma, \eta) = \inf_n \left[ \frac{(1-\eta)\gamma (n-1) + \chi (n\sigma + 1-\gamma)}{1 + \frac{(n-1)(\lambda - a)\beta + n\delta}{\mu + \lambda}} \right]^{1/\eta} \]

(\( n = 2, 3, \ldots \)).

\[ h_2(\sigma, \chi, \gamma, \eta) = \inf_n \left[ \frac{(1-\eta)\gamma (n-1) + \chi (n\sigma + 1-\gamma)}{1 + \frac{(n-1)(\lambda - a)\beta + n\delta}{\mu + \lambda}} \right]^{1/\eta} \]
Definition 4.2 The fractional derivative of order $u$ is defined by

$$
D_z^u f(z) = \frac{1}{\Gamma(1-u)} \int_0^z \frac{f(t)}{(z-t)^{1-u}} dt,
$$

where, $0 \leq u < 1$, $f(z)$ is analytic function in a simply connected region of the $z$-plane containing the origin and multiplicity of $(z-t)^{-u}$ is remove as in Definition 4.1.

Definition 4.3 Under the conditions of Definition 4.2, the fractional derivative of order $n+u$ is defined by

$$
D_z^n f(z) = \frac{d^n}{dz^n} D_z^u f(z),
$$

where $0 \leq u < 1$ and $n = 0, 1, 2, \ldots$.

Theorem 4.4 Let the function $f(z)$ be in the class $G_m(\gamma, \sigma, \lambda, \mu)$. Then

$$
|D_z^{-\gamma} f(z)| \leq \frac{1}{\Gamma(2+\gamma)} |z|^{1+\gamma}
$$

(13)

And

$$
|D_z^{-\gamma} f(z)| \leq \frac{1}{\Gamma(2+\gamma)} |z|^{1+\gamma}
$$

(14)

The last equalities in (13) and (14) are accomplished for the function

$$
f(z) = z - \frac{\overline{\chi} \left(\sigma + (1-\gamma)\right)}{\chi \left(1+\gamma - (2\sigma + 1-\gamma)\right) + 1+ \left[\overline{\lambda} \left(\gamma + (2\sigma + 1-\gamma)\right)\right]^{\mu+\lambda}} z^2.
$$
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**Theorem 4.5** Let the function $f(z)$ be in the class $G_m(γ, σ, χ)$. Then,

$$|D_z^v f(z)| \leq \frac{1}{\Gamma(2-v)} |z|^{1-v}$$

And

$$|D_z^v f(z)| \geq \frac{1}{\Gamma(2-v)} |z|^{1-v}$$

where

$$\chi(σ + (1-γ)) \left[ γ + χ(2σ + 1-γ) \right] \left[ 1 + \frac{(λ-α)β + 2δ}{μ + λ} \right]^m$$

The equalities in (17) and (18) are accomplished for a univalent function

$$f(z) = z - \frac{χ(σ + (1-γ))}{γ + χ(2σ + 1-γ)} \frac{1}{1 + \frac{(λ-α)β + 2δ}{μ + λ}} z^2$$

Proof. By use the first important result coefficient ©in equality in Theorem 2.1

$$\sum_{n=2}^{∞} a_n \leq \frac{χ(σ + (1-γ))}{γ + χ(2σ + 1-γ)} \left[ 1 + \frac{(λ-α)β + 2δ}{μ + λ} \right]^m \sum_{n=2}^{∞} \frac{Γ(n+1)}{(n+ν+1)} a_n z^{n+ν}$$

And

$$\Gamma(2+v) z^{-ν} D_z^{-v} f(z) = z - \sum_{n=2}^{∞} \frac{Γ(n+1)}{(n+ν+1)} a_n z^n = z - \sum_{n=2}^{∞} \Psi(n) a_n z^n$$

where

$$\Psi(n) = \frac{Γ(n+1)Γ(v+2)}{Γ(n+ν+1)}.$$

We get that $\Psi(n)$ is a decreasing for a univalent function of $n$ and $0 < \Psi(n) \leq \Psi(2) = \frac{2}{2+v}$. Using (15) and (16) we get

$$\left| \Gamma(2+v) z^{-ν} D_z^{-v} f(z) \right| \leq |z| + \Psi(2)|z|^2 \sum_{n=2}^{∞} a_n$$

$$\leq |z| + \frac{2χ(σ + (1-γ))}{(2+v)(γ + χ(2σ + 1-γ))} \left[ 1 + \frac{(λ-α)β + 2δ}{μ + λ} \right]^m |z|^2,$$

which gives (13), we likewise have

$$\left| \Gamma(2+v) z^{-ν} D_z^{-v} f(z) \right| \geq |z| - \Psi(2)|z|^2 \sum_{n=2}^{∞} a_n$$

$$\geq |z| - \frac{2χ(σ + (1-γ))}{(2+v)(γ + χ(2σ + 1-γ))} \left[ 1 + \frac{(λ-α)β + 2δ}{μ + λ} \right]^m |z|^2,$$

which gives (14).
From Definition 4.2 we get
\[ D_z^\nu f(z) = \frac{1}{\Gamma(2-\nu)} z^{1-\nu} - \sum_{n=2}^\infty \frac{\Gamma(n+1)}{\Gamma(n-\nu+1)} a_n z^{n-\nu} \]

And
\[ \Gamma(2-\nu)z^\nu D_z^\nu f(z) = \left[ z - \sum_{n=2}^\infty \frac{\Gamma(n)\Gamma(2-\nu)}{\Gamma(n-\nu+1)} n a_n z^n \right] - \sum_{n=2}^\infty n \Phi(n) a_n z^n \]
\[ = z - \sum_{n=2}^\infty \frac{\Gamma(n)\Gamma(2-\nu)}{\Gamma(n-\nu+1)} n a_n z^n = z - \sum_{n=2}^\infty n \Phi(n) a_n z^n \]
(20)

since \( \Phi(n) = \frac{\Gamma(n)\Gamma(2-\nu)}{\Gamma(n-\nu+1)} \)

We know that \( \Phi(n) \) is a decreasing for a univalent function of \( n \) and \( 0 < \Phi(n) \leq \Phi(2) = \frac{2}{2-\nu} \)

Using (19) and (20) we have
\[ |(2-\nu) z^\nu D_z^\nu f(z)| \leq |z| + |\Phi(2)| + \sum_{n=2}^\infty n a_n \leq 2^{\nu-1} \left[ 2 \chi \left( \sigma + (1-\gamma) \right) \left[ 1 + \frac{\left( \alpha - a \right) \beta + 2\delta}{\mu + \lambda} \right] \right]^{\infty} |z| \]
\[ \left[ (2-\nu) \left[ \gamma + \chi \left( 2\sigma + 1-\gamma \right) \right] \left[ 1 + \frac{\left( \alpha - a \right) \beta + 2\delta}{\mu + \lambda} \right] \right] \]

which gives (17), we also have
\[ \left| D_z^\nu f(z) \right| \leq |z| + |\Phi(2)| + \sum_{n=2}^\infty n a_n \]
\[ \leq |z| \left[ 2 \chi \left( \sigma + (1-\gamma) \right) \left[ 1 + \frac{\left( \alpha - a \right) \beta + 2\delta}{\mu + \lambda} \right] \right]^{\infty} |z| \]
\[ \left[ (2-\nu) \left[ \gamma + \chi \left( 2\sigma + 1-\gamma \right) \right] \left[ 1 + \frac{\left( \alpha - a \right) \beta + 2\delta}{\mu + \lambda} \right] \right] \]
which gives (18).

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6. References