Fuzzy Translation and Fuzzy Multiplication in BF/BG-algebras

M. Chandramouleeswaran¹, P. Muralikrishna²* and S. Srinivasan²

¹Associate Professor and Head, Department of Mathematics, SBK College, Aruppukottai-626 101, Tamilnadu; moulee59@gmail.com
²Assistant Professor, School of Advanced Sciences, VIT University, Vellore-632 014, Tamilnadu; pmkrishna@rocketmail.com; smrail@gmail.com

Abstract
This paper deals with the notion of fuzzy translation and fuzzy multiplication on BF-algebras and investigates some of their basic properties.

Keywords: BF-algebra, BG-algebra, Fuzzy BF-subalgebra, Fuzzy Translation, Fuzzy Multiplication, Fuzzy Magnified Translation.

AMS Subject Classification
08A72, 03E72, 03F55, 03G25.

1. Introduction
Imai and Iseki [3] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Neggers and Kim [7] introduced the notion of B-algebras, which is a generalization of BCK-algebra. During 1998, Jun et al. [4] introduced BH-algebras, which are a generalization of BCK/BCI/B-algebras. The study of BF-algebra was initiated by Walendziak [1].

The notion of a fuzzy subset was initially introduced by Zadeh [8] in 1965, for representing uncertainty. With these ideas, fuzzy BF-subalgebras of were developed by Saeid and Rezvani [2]. The concept of Fuzzy Translation is applied in BCK/BCI algebras, by Lee and Jun [5]. Motivated by these in this paper the notion of fuzzy translation on BF-algebras has been introduced.

2. Preliminaries
In this section the basic definitions of a BF-algebra, subalgebra, fuzzy subset and fuzzy BF-subalgebra are recalled. We start with,

*Corresponding author:
P. Muralikrishna (pmkrishna@rocketmail.com)
**Example 2.4:** The set $X = \{0, 1, 2\}$ and $*$ defined by the following table is a BF-algebra.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
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<tr>
<td>1</td>
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<td>0</td>
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</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

**Definition 2.5:** A binary relation “≤” on an BF-algebra $X$ is defined as $x \leq y$ if and only if $x * y = 0$.

**Definition 2.6:** [1] A non-empty subset $S$ of a BF-algebra $X$ is said to be a subalgebra if

$x * y \in S \quad \forall x, y \in S$.

**Definition 2.7:** [8] A fuzzy subset $\mu$ in a non-empty set $X$ is a function $\mu : X \rightarrow [0,1]$.

**Definition 2.8:** [2] A Fuzzy Subset $\mu$ in a BF-algebra $X$ is said to be a Fuzzy BF-subalgebra of $X$ if $\mu(x * y) \geq \min{\mu(x), \mu(y)} \quad \forall x, y \in X$.

**Example 2.9:** Consider the BF-algebra $X = \{0, 1, 2, 3, 4\}$ in Example 2.2. Define a fuzzy subset $\mu$ of $X$ by

$\mu(x) = \begin{cases} 
0.6 & x \neq 2 \\
0.1 & x = 2
\end{cases}$

Then $\mu$ is fuzzy BF-subalgebra of $X$.

**3. Fuzzy Translation and Fuzzy Multiplication on a BF-algebra**

This section deals with the notion of Fuzzy translation and Fuzzy multiplication on BF-algebras. In what follows, $X$ denotes a BF-algebra, and for any fuzzy set $\mu$ of $X$, we denote $T = \sup{\mu(x) : x \in X}$ unless otherwise specified. We start with,

**Definition 3.1:** [5, 6] Let $\mu$ be a fuzzy subset of $X$ and $\alpha \in [0, T]$. A mapping $\mu^\alpha : X \rightarrow [0,1]$ is said to be a fuzzy-$\alpha$-translation of $\mu$ if it satisfies $\mu^\alpha(x) = \mu(x) + \alpha \quad \forall x \in X$.

**Definition 3.2:** [5, 6] Let $\mu$ be a fuzzy subset of $X$ and $\alpha \in [0, T]$. A mapping $\mu^\alpha : X \rightarrow [0,1]$ is said to be a fuzzy-$\alpha$-multiplication of $\mu$ if it satisfies $\mu^\alpha(x) = \alpha \cdot \mu(x) \quad \forall x \in X$.

**Example 3.3:** Consider the BF-algebra $X = \{0, 1, 2, 3, 4\}$ in Example 2.2. Define a fuzzy subset $\mu$ of $X$ by

$\mu(x) = \begin{cases} 
0.4 & x \neq 2 \\
0.1 & x = 2
\end{cases}$

Then $\mu$ is fuzzy BF-subalgebra of $X$.

Then the mapping $\mu^\alpha : X \rightarrow [0,1]$ defined by

$\mu^\alpha(x) = \begin{cases} 
0.4 + 0.3 = 0.7 & x \neq 2 \\
0.1 + 0.3 = 0.4 & x = 2
\end{cases}$

which satisfies $\mu^\alpha(x) = \mu(x) + 0.3 \quad \forall x \in X$, is a fuzzy-0.3-translation.

And the mapping $\mu^\alpha : X \rightarrow [0,1]$ defined by

$\mu^\alpha(x) = \begin{cases} 
0.2(0.4) = 0.08 & x \neq 2 \\
0.2(0.1) = 0.02 & x = 2
\end{cases}$

which satisfies $\mu^\alpha(x) = 0.2 \cdot \mu(x) \quad \forall x \in X$, is fuzzy-0.2-multiplication.

**Theorem 3.4:** For any fuzzy BF-subalgebra $\mu$ of $X$ and $\alpha \in [0, T]$, the fuzzy-$\alpha$-translation $\mu^\alpha(x)$ of $\mu$ is a fuzzy BF-subalgebra of $X$.

**Proof:** Let $x, y \in X$ and $\alpha \in [0, T]$.

Then $\mu(x * y) \geq \min{\mu(x), \mu(y)}$.

Now $\mu^\alpha(x * y) = \mu(x * y) + \alpha$

$\geq \min{\mu(x), \mu(y)} + \alpha$

$= \min{\mu(x + \alpha, \mu(y) + \alpha} = \min{\mu^\alpha(x), \mu^\alpha(y)}$

This completes the proof.

The following is the converse of the above theorem.

**Theorem 3.5:** For any fuzzy subset $\mu$ of $X$ and $\alpha \in [0, T]$, if the fuzzy-$\alpha$-translation $\mu^\alpha(x)$ of $\mu$ is a fuzzy BF-subalgebra of $X$, then so is $\mu$.

**Proof:** Let $x, y \in X$.

Assume that $\mu^\alpha(x)$ of $\mu$ is a fuzzy BF-subalgebra of $X$ for $\alpha \in [0, T]$.

Then $\mu(x * y) + \alpha = \mu^\alpha(x * y)$

$\geq \min{\mu^\alpha(x), \mu^\alpha(y)}$

$= \min{\mu(x) + \alpha, \mu(y) + \alpha} = \min{\mu(x), \mu(y)} + \alpha$

$\Rightarrow \mu(x * y) \geq \min{\mu(x), \mu(y)}$

This completes the proof.

**Theorem 3.6:** For any fuzzy BF-subalgebra $\mu$ of $X$ and $\alpha \in [0, T]$, the fuzzy-$\alpha$-multiplication $\mu^\alpha(x)$ of $\mu$ is a fuzzy BF-subalgebra of $X$.

**Proof:** Let $x, y \in X$ and $\alpha \in [0, T]$. 

\[ \text{Choose } \alpha = 0.3, \beta = 0.2 \in [0,1] \]
Then $\mu(x \ast y) \geq \min \{\mu(x), \mu(y)\}$

Now $\mu^M_a(x \ast y) = \alpha \cdot \mu(x \ast y)$

$\geq \alpha \cdot \min \{\mu(x), \mu(y)\}$

$= \min \{\alpha \cdot \mu(x), \alpha \cdot \mu(y)\}$

$= \min (\mu^M_a(x), \mu^M_a(y))$

This completes the proof.

The following is the converse of the above theorem.

**Theorem 3.7:** For any fuzzy subset $\mu$ of $X$ and $\alpha \in [0,1]$, if the fuzzy-$\alpha$-multiplication $\mu^T_a(x)$ of $\mu$ is a fuzzy BF-subalgebra of $X$ then so is $\mu$.

**Proof:** Let $x, y \in X$.

Assume that $\mu^T_a(x)$ of $\mu$ is a fuzzy BF-subalgebra of $X$ for $\alpha \in [0,1]$.

Then $\alpha \cdot \mu(x \ast y) = \mu^M_a(x \ast y)$

$\geq \min \{\mu^M_a(x), \mu^M_a(y)\}$

$= \min \{\alpha \cdot \mu(x), \alpha \cdot \mu(y)\}$

$= \alpha \cdot \min \{\mu(x), \mu(y)\}$

$\Rightarrow \mu(x \ast y) \geq \min \{\mu(x), \mu(y)\}$

This completes the proof.

**Definition 3.8.** [6] Let $\mu$ be a fuzzy subset of $X$, $\alpha \in [0,1]$ and $\beta \in [0,1]$. A mapping $\mu^M_{\beta \alpha} : X \rightarrow [0,1]$ is said to be a fuzzy magnified-$\beta \alpha$-translation of $\mu$ if it satisfies $\mu^M_{\beta \alpha}(x) = \beta \cdot \mu(x) + \alpha \ \forall \ x \in X$.

**Example 3.9:** Consider the BF-algebra $X = \{0,1,2,3,4\}$ in Example 2.2. Define a fuzzy subset $\mu$ of $X$ by

$\mu(x) = \begin{cases} 0.4 & x \neq 2 \\ 0.2 & x = 2 \end{cases}$. Then $\mu$ is fuzzy BF-subalgebra of $X$.

Here $T = 1 - \sup \{\mu(x) : x \in X\} = 1 - 0.4 = 0.6$. Choose $\alpha = 0.2 \in [0,1]$ and $\beta = 0.3 \in [0,1]$.

Then the mapping $\mu^M_{(0.2,0.3)} : X \rightarrow [0,1]$ defined by

$\mu^M_{(0.2,0.3)}(x) = \begin{cases} (0.3)(0.4) + 0.2 = 0.32 & x \neq 2 \\ (0.3)(0.2) + 0.2 = 0.26 & x = 2 \end{cases}$, which satisfies $\mu^M_{(0,0,0,0)}(x) = (0.3) \cdot \mu(x) + 0.2 \ \forall \ x \in X$ is fuzzy magnified $(0.3)(0.2)$-translation.

**Theorem 3.10:** Let $\mu$ be a fuzzy subset of $X$, $\alpha \in [0,1]$ and $\beta \in [0,1]$. A mapping $\mu^M_{\beta \alpha} : X \rightarrow [0,1]$ is said to be a fuzzy magnified-$\beta \alpha$-translation of $\mu$. Then $\mu$ is fuzzy BF-subalgebra of $X$ if and only if $\mu^M_{\beta \alpha}$ is fuzzy subalgebra of $X$.

**Proof:** Let $\mu$ be a fuzzy subset of $X$, $\alpha \in [0,1]$ and $\beta \in [0,1]$. A mapping $\mu^M_{\beta \alpha} : X \rightarrow [0,1]$ is said to be a fuzzy magnified-$\beta \alpha$-translation of $\mu$.

Assume $\mu$ is fuzzy BF-subalgebra of $X$.

Then $\mu(x \ast y) \geq \min \{\mu(x), \mu(y)\}$

Now $\mu^M_{\beta \alpha}(x \ast y) = \beta \cdot \mu(x \ast y) + \alpha$

$\geq \beta \cdot \{\min \{\mu(x), \mu(y)\}\} + \alpha$

$= \min \{\beta \cdot \{\mu(x) + \alpha\}, \beta \cdot \{\mu(y) + \alpha\}\}$

$= \min (\mu^M_{\beta \alpha}(x), \mu^M_{\beta \alpha}(y))$

Hence $\mu^M_{\beta \alpha}$ is fuzzy subalgebra of $X$.

Conversely assume $\mu^M_{\beta \alpha}$ is fuzzy subalgebra of $X$.

Then $\beta \cdot \mu(x \ast y) + \alpha = \mu^M_{\beta \alpha}(x \ast y)$

$\geq \min (\mu^M_{\beta \alpha}(x), \mu^M_{\beta \alpha}(y))$

$= \min \{\beta \cdot \{\mu(x) + \alpha\}, \beta \cdot \{\mu(y) + \alpha\}\}$

$= \beta \cdot \{\min \{\mu(x), \mu(y)\}\} + \alpha$

$\Rightarrow \mu(x \ast y) \geq \min \{\mu(x), \mu(y)\}$

Hence $\mu$ is fuzzy BF-subalgebra of $X$.

This completes the proof.

**4. Conclusion**

In this article we have introduced the notion of fuzzy translation and fuzzy multiplication on BF-algebras. Surprisingly Walendziak [1] Theorem 2.11 says that the structure BF-algebra becomes a BG-algebra, the proof following directly from the definition. Hence we conclude that whatever result we have proved for BF-algebras can directly be carried over to BG-algebras.
5. References