Perturbations of Generalized Dual Banach Frames

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Abstract

The classical duals and generalized duals of frames play a fundamental role in the abstract frame theory. In this paper we first define the concepts of dual, canonical dual, pseudo-dual and approximate dual for Bessel sequences with respect to a BK-space of scalar-valued sequences in Banach spaces, and illuminate their relationship with Banach frames. Then we prove that classical dual and approximate dual of Banach frames are stable under small perturbations so that the results obtained is a special case of it. We also apply a new characterization of classical dual Banach frames to discuss a stability problem for them. For approximate dual Banach frames constructed via perturbation theory, we provide a bound on the deviation from perfect reconstruction.

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1. Introduction

$\ell^2$-Frames in Hilbert spaces were first introduced by Duffin and Schaeffer in the study of nonharmonic Fourier series in 1952 and then were applied by Daubechies et al. in the wavelet and Gabor transform. On the other hand, Grochenig extended $\ell^p$-frames to Banach frames with respect to a BK-space of scalar-valued sequences in Banach spaces. Further $\ell^2$-frames also were introduced by Aldroubi et al. and Christensen et al. as a tool to obtain series expansions in shift-invariant spaces. The theory of $\ell^2$-frames in Hilbert spaces presents a central tool in many areas and has developed rather rapidly in the past decade. Dual Banach frames pair are sequences in a Banach space and its dual that have basis-like properties but which need not be basis. In particular, they allow elements of a Banach space to be written as linear combinations of the Banach frame elements. Unfortunately, it is usually complicated to calculate a dual Banach frame explicitly. Hence we seek methods for constructing generalized duals. Approximate dual and pseudo-dual frames in Hilbert spaces are defined by Christensen in. The main purpose of this paper is to further and perfect study of the concepts of pseudo-dual and approximate dual Banach frames and examines their properties. We also investigate the use of perturbation theory to construct pseudo-dual and approximate dual Banach frames.

In the rest of this introduction we will briefly recall the definitions and basic properties of Banach frames and bases. For more information we refer to the works of Casazza. Then, in Section 2 we discuss dual Banach frames and find some characterizations about them.

In Section 3 we introduce the concepts of pseudo-dual and approximate dual Banach frames and we show that these concepts are stable under small perturbations. We introduce new and weaker conditions which ensure the desired stability.

Throughout the paper $X$ will be a separable Banach space and $I$ is a countable index set that has been well-ordered. We shall denote by $I_n, n \in \mathbb{N}$, the subset of the first $n$ indices in $I$. If $|I| < \infty$ then $I_n = I$ for all $n \geq |I|$. The notation $X_d$ will always be reserved to denote a BK-space on $I$.

In a BK-space the canonical unit vectors are the elements $e_i$ defined by $e_i(j) = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker

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As the proof of Proposition 2.3 we can prove that \( V : X_d \to X \) is also a bounded operator. We call \( \{ x_i \} \) a Banach frame for \( X^* \) with respect to \( X_d^* \). If it is a \( X_d^* \)-frame for \( X^* \) and there exists a bounded linear operator \( S_i : X \to X_d \) such that \( V S_i x = x \), for all \( x \in X \). Similarly the mapping \( S_i \) is said the reconstruction operator and the optimal Banach frame bounds are \( ||S||_1, ||V|| \).

We say that a family \( \{ f_i \} \subseteq X^* \) is total in \( X^* \), if \( f_i(x) = 0 \), for all \( i \in I \) then \( x = 0 \). Similarly a sequence \( \{ x_i \} \subseteq X \) is said to be total, if \( f_i(x_i) = 0 \), for all \( i \in I \) then \( f = 0 \). The following example shows that in a Hilbert space, there is a Banach frame with respect to a BK-space which is not a Banach frame with respect to \( \ell^p(N) \).

**Example 1.2:** Suppose that \( \{ e_i \} \) is an orthonormal basis for a separable Hilbert space \( H \). Consider the family \( \{ e + e_{im} \} \), which is a complete set in \( H \). Then by Lemma 2.6, it is a Banach frame with respect to the BK-space

\[
X_d = \left\{ \left\{ (h, e + e_{im}) \right\}_h \in H \right\}
\]

but not a Banach frame for \( H \) with respect to \( \ell^p(N) \).

**Definition 1.3:** Let \( \{ x_i \} \subseteq X \) be a sequence in \( X \). Then

(i) \( \{ x_i \} \) is called a Schauder basis for \( X \) if for every \( x \in X \) there is a unique sequence of scalars \( \{ c_i \} \) which called the coordinates of \( x \), such that \( x = \sum c_i x_i \).

(ii) \( \{ x_i \} \) is said to be a \( X_d \)-Riesz basis for \( X \) if it is a total set in \( X \) and there exist two positive constants \( 0 < A < B < \infty \) such that,

\[
A \| x \|_X^* \leq \left| \sum c_i x_i \right|_X \leq \| c \|_{X_d} \quad \forall c \in X_d.
\]
\[ X_d^* = \{ \{ f(x_i) \}_{i \in I} \mid f \in X' \} \]

which its norm given by \( \| \{(f(x_i))_{i \in I}\} \| = \| |x| \cdot \| X \| \). The above definitions show that every \( x \in X \) has a unique expansion of the form \( x = \sum_{i \in I} f(x_i) x_i \) and \( \{ f_i \}_{i \in I}, \{ x_i \}_{i \in I} \) are Banach frames for \( X, X^* \) with respect to respectively \( X_d, X_d^* \). Since \( \| \sum_{i \in I} f_i(x_i) x_i = \| x \| = \| \{ f_i(x_i) \}_{i \in I} \| \), hence \( \{ x_i \}_{i \in I} \) is also a \( X_d \)-Riesz basis for \( X \).

A sequence \( \{ x_i \}_{i \in I} \) is called minimal in \( X \) if for every 
\[ i \in I, x_i \not\in \text{span}\{x_i \}_{i \in I}. \]

The following Propositions are important to characterising \( X_d^* \)-frames amid \( X_d \)-Riesz bases which were proved.

**Proposition 1.4:** Let \( \{ x_i \}_{i \in I} \subseteq X \) and \( \{ f_i \}_{i \in I} \subseteq X' \) then

(i) \( \{ f_i \}_{i \in I} \) is a \( X_d \)-Bessel sequence for \( X \) with \( X_d \)-Bessel bound \( B \) if and only if \( \sum_{i \in I} d_i f_i \) converges in \( X^* \) for all \( d \in X_d \) and \( \| \sum_{i \in I} d_i f_i \| \leq B \| d \| \).

(ii) \( \{ x_i \}_{i \in I} \) is a \( X_d^* \)-Bessel sequence for \( X^* \) with \( X_d^* \)-Bessel bound \( B \) if and only if \( \sum_{i \in I} \epsilon_i x_i \) converges in \( X \) for all \( \epsilon \in X_d \) and \( \| \sum_{i \in I} \epsilon_i x_i \| \leq B \| \epsilon \| \).

(iii) \( \{ x_i \}_{i \in I} \) is a \( X_d \)-frame for \( X \) if and only if, the synthesis operator \( V : X_d \rightarrow X^* \), is bounded and possess a bounded inverse on \( R_d \).

**Proposition 1.5:** The sequence \( \{ x_i \}_{i \in I} \) is a \( X_d \)-Riesz basis with Riesz bounds \( A, B \) for \( X \) if and only if \( \{ x_i \}_{i \in I} \) is minimal in \( X \) and a \( X_d \)-frame for \( X \) with same bounds \( A, B \).

## 2. Dual Banach Frames

We start this section by definition of dual Banach frames and proving some characterizations about them. Then we discuss a perturbation property for dual Banach frames.

**Definition 2.1:** Let \( \{ f_i \}_{i \in I} \subseteq X^* \), \( \{ x_i \}_{i \in I} \subseteq X \) be \( X_d \)-Bessel and \( X_d^* \)-Bessel sequences for \( X, X^* \) respectively. Then

(i) \( \{ f_i \}_{i \in I} \) is called a dual Banach frame for \( \{ x_i \}_{i \in I} \) in \( X \) with respect to \( X_d \), if \( x = \sum_{i \in I} f_i(x_i) x_i \) for all \( x \in X \), with the norm convergent sense in \( X \).

(ii) \( \{ x_i \}_{i \in I} \) is called a dual Banach frame for \( \{ f_i \}_{i \in I} \) in \( X^* \) with respect to \( X_d^* \), if \( f = \sum_{i \in I} f_i(x_i) f_i \) for all \( f \in X^* \), with the norm convergent sense in \( X^* \).

We begin with an equivalence result of duality on the \( X_d \)-Bessel and \( X_d^* \)-Bessel sequences respectively.

**Lemma 2.2:** Let \( \{ f_i \}_{i \in I}, \{ x_i \}_{i \in I} \) be \( X_d \)-Bessel and \( X^*_d \)-Bessel sequences for \( X, X^* \) respectively.

Then the following statements are equivalent:

(i) \( \{ f_i \}_{i \in I} \) is a dual Banach frame for \( \{ x_i \}_{i \in I} \) in \( X \) with respect to \( X_d \).

(ii) \( \{ x_i \}_{i \in I} \) is a dual Banach frame for \( \{ f_i \}_{i \in I} \) in \( X^* \) with respect to \( X^*_d \).

(iii) \( f(x) = \sum_{i \in I} f_i(x_i) f_i(x) \) for all \( x \in X, f \in X^* \).

**Proof:** To prove (i) \( \Rightarrow \) (ii) let \( f \in X^* \) be arbitrary then by Proposition 1.4 the series \( \sum_{i \in I} f(x_i) f_i \) is convergent in \( X^* \) and for every \( x \in X \) we have:

\[ f(x) = f\left( \sum_{i \in I} f_i(x_i) x_i \right) = \sum_{i \in I} f(x_i) f_i(x) = \left( \sum_{i \in I} f_i(x_i) f_i \right)(x) \]

This shows that \( f = \sum_{i \in I} f_i(x_i) f_i \). The implications (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (i) are obvious.

Therefore, a pair of \( X_d \)-Bessel and \( X_d^* \)-Bessel sequences \( \{ f_i \}_{i \in I}, \{ x_i \}_{i \in I} \) is called a dual Banach frame pair for \( X \) or \( X^* \), if one of the conditions in Lemma 2.2 is satisfied.

**Lemma 2.3:** Let \( \{ f_i \}_{i \in I}, \{ x_i \}_{i \in I} \) be a dual Banach frame pair for \( X \) and \( X^* \), with respect to \( X_d, X_d^* \) respectively.

**Proof:** In terms of the analysis and synthesis operators of \( \{ f_i \}_{i \in I}, \{ x_i \}_{i \in I} \). The assumption assures that \( U^* V(f) = f \) for all \( f \in X^* \). This follows that

\[ \| f \| = \| U^* V(f) \| \leq \| U \| \| V(f) \| \leq \| U \| \| \{ f(x_i) \}_{i \in I} \| \]

Now from \( VU(x) = x \) for all \( x \in X \) implies that \( \{ x_i \}_{i \in I} \) is a Banach frame for \( X \) with respect to \( X_d \). Similarly, we can show that \( \{ f_i \}_{i \in I} \) is also a Banach frame for \( X \) with respect to \( X_d^* \).

The following proposition will show that a Banach frame for \( X \) with respect to \( X_d \) plays the same role in Banach frame theory as the dual in the theory of bases.

**Proposition 2.4:** Let \( \{ f_i \}_{i \in I} \) be a Banach frame for \( X \) with respect to \( X_d \), then there exists a \( X_d^* \)-Bessel sequence...
\[ \{ \tilde{x}_i \}_{i=1}^n \] for \( X^* \) such that \( \{ f_i \}_{i=1}^n \) is a dual Banach frame for \( \{ \tilde{x}_i \}_{i=1}^n \) in \( X \) with respect to \( X_d \).

**Proof:** Since \( \{ f_i \}_{i=1}^n \) is a Banach frame for \( X \) with respect to \( X_d \), hence there exists a bounded operator \( S_U : X_d \to X \) such that \( S_U x = f \) where, \( U : X \to X_d \) is the analysis operator of \( \{ f_i \}_{i=1}^n \). Put \( \tilde{x}_i = S_i(e_i) \) where, \( e_i, i \in I \) is the Schauder basis of the canonical unit vectors in \( X_d \). We first show that \( \{ \tilde{x}_i \} \) is a \( X_d \) Bessel sequence for \( X^* \). Given \( c \in X_d \) and \( m, n \in \mathbb{N} \) with \( m > n \),

\[
\| \| \sum_{i=1}^{n} c_i \tilde{x}_i \|_X = \| \sum_{i=1}^{n} \| c_i S_i(e_i) \|_X \\
= \| S_U \left( \sum_{i=1}^{n} c_i \tilde{x}_i \right) \|_X \leq \| S_U \| \sum_{i=1}^{n} \| c_i e_i \|_X.
\]

Since \( c \in X_d \) hence \( S_U e_i \) is convergent, this implies that \( \{ \sum_{i=1}^{n} c_i \tilde{x}_i \}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( X \) and therefore convergent. By Proposition 1.4, \( \{ \tilde{x}_i \} \) is a \( X_d \) Bessel sequence for \( X^* \). Moreover, for every \( x \in X \) we have

\[
x = S_U(x) = S_U \left( \sum_{i=1}^{n} f_i(x) e_i \right) = \sum_{i=1}^{n} f_i(x) S_i(e_i) = \sum_{i=1}^{n} f_i(x) \tilde{x}_i.
\]

This shows that \( \{ f_i \}_{i=1}^n \) is a dual Banach frame for \( \{ \tilde{x}_i \}_{i=1}^n \) in \( X \) with respect to \( S_U \).

Via Lemma 2.2 the Banach frame \( \{ \tilde{x}_i \}_{i=1}^n \) is said to be the canonical dual Banach frame of \( \{ f_i \}_{i=1}^n \) in \( X^* \) with respect to \( X_d \). We also have a parallel result for Banach frames for \( X^* \) with respect to \( X_d \).

**Theorem 2.5:** Let \( \{ x_i \}_{i=1}^n \) be a Banach frame for \( X^* \) with respect to \( X_d \) then there exists a \( X_d \) Bessel sequence \( \{ f_i \}_{i=1}^n \) for \( X \) such that \( \{ f_i \}_{i=1}^n \) is a dual Banach frame for \( \{ x_i \}_{i=1}^n \) in \( X^* \) with respect to \( X_d \).

**Proof:** By the assumption there exists a bounded operator \( S : X \to X_d \) such that \( VS = I_X \) where, \( V : X \to X \) is the synthesis operator of \( \{ x_i \}_{i=1}^n \). Put \( f_i = S_i(e_i) \) where \( e_i \) is the Schauder basis of the canonical unit vectors in \( X_d \). Since for all \( x \in X \) we have:

\[
\| \{ f_i(x) \}_{i=1}^n \|_{X_d} = \| \{ S_i(e_i(x)) \}_{i=1}^n \|_{X_d} \\
= \| \{ e_i(S_i(x)) \}_{i=1}^n \|_{X_d} \\
= \| S_i(x) \|_{X_d} \leq \| S_i \| \| x \|_X,
\]

\[ \text{hence } \{ f_i(x) \}_{i=1}^n \text{ is a } X_d \text{ Bessel sequence for } X. \text{ Moreover, for all } f \in X^* \text{ we have}
\]

\[ f = S_i V (f) = S_i \left( \sum_{i=1}^{n} f_i(x) e_i \right) = \sum_{i=1}^{n} f_i(x) S_i(e_i) = \sum_{i=1}^{n} f_i(x) f_i.
\]

From this the result follows.

The Banach frame \( \{ x_i \}_{i=1}^n \) is called the canonical dual Banach frame of \( \{ x_i \}_{i=1}^n \) in \( X \) with respect to \( X_d \). The next theorem generalizes a result of Christensen \( ^{11} \) to the situation of dual Banach frames. In this theorem we show that every Banach frame has infinitely many dual Banach frames.

**Theorem 2.6:** Let \( \{ x_i \}_{i=1}^n \subseteq X \) and \( \{ f_i \}_{i=1}^n \subseteq X^* \). Then the following holds:

(i) If \( \{ f_i \}_{i=1}^n \) is a Banach frame for \( X \) with respect to \( X_d \) with the analysis operator \( U \). Then the dual Banach frames for \( \{ f_i \}_{i=1}^n \) in \( X^* \) with respect to \( X_d \) are precisely the families \( \{ x_i \}_{i=1}^n = \{ T_i e_i \}_{i=1}^n \), where, \( T_i : X_d \to X \) is a bounded left-inverse of \( U \).

(ii) If \( \{ x_i \}_{i=1}^n \) is a Banach frame for \( X^* \) with respect to \( X_d \) with the synthesis operator \( V \). Then the dual Banach frames for \( \{ x_i \}_{i=1}^n \) in \( X \) with respect to \( X_d \) are precisely the families \( \{ f_i \}_{i=1}^n = \{ T'_i e_i \}_{i=1}^n \), where, \( T'_i : X \to X_d \) is a bounded right-inverse of \( V \).

**Proof:** The proof is identical to the proof of Propositions 2.4 and 2.5.

The next theorem is analogous to Lemma 5.7.3 \( ^{11} \) to the situation of dual Banach frames.

**Theorem 2.7:** Let \( \{ f_i \}_{i=1}^n \), \( \{ x_i \}_{i=1}^n \) be Banach frames for \( X, X^* \) with respect to \( X_d, X_d' \) with the analysis and synthesis operators \( U, V \), respectively. Then the following holds:

(i) The bounded left-inverses of \( U \) are precisely the operators having the form \( S_i + W (I_{X_d} - US) \) where, \( W : X \to X_d \) is a bounded operator and \( S_i \) denotes the reconstruction operator of \( \{ f_i \}_{i=1}^n \).

(ii) The bounded right-inverses of \( V \) are precisely the operators having the form \( S_i + (I_{X_d} - S_i V) W \) where, \( W : X \to X_d \) is a bounded operator and \( S_i \) denotes the reconstruction operator of \( \{ x_i \}_{i=1}^n \).

**Proof:** For the proof of (i), it is obvious that an operator of the given form is a left-inverse of \( U \). On the other hand, if \( T_i \) is a given left-inverse of \( U \), then by taking \( W = T_i \) we have \( T_i = S_i + T_i (I_{X_d} - US) \).
the argument for statement (ii) is similar.

The next theorem is analogous to a well-known result in abstract frame theory Theorem 5.7.4. This theorem is a characterization of all dual Banach frames associated with a given Banach frame.

**Theorem 2.8:** Let \( \{f_i\}_{i=1}^\infty \) and \( \{x_i\}_{i=1}^\infty \) be Banach frames for \( X, X' \) with respect to \( X_p', X_p \), respectively. Then the following holds:

(i) The dual Banach frames of \( \{f_i\}_{i=1}^\infty \) in \( X' \) with respect to \( X_d' \) are precisely the families

\[
\{z_k\}_{k=1}^\infty = \left\{ \hat{x}_k + y_k - \sum_{i=1}^\infty f_i(\hat{x}_i)y_i \right\}_{k=1}^\infty,
\]

where, \( \{y_k\}_{k=1}^\infty \) is a \( X_d' \)-Bessel sequence for \( X' \) and \( \{x_i\}_{i=1}^\infty \) denotes the canonical dual Banach frame of \( \{f_i\}_{i=1}^\infty \) in \( X' \) with respect to \( X_d' \).

(ii) The dual Banach frames of \( \{x_i\}_{i=1}^\infty \) in \( X \) with respect to \( X_d \) are precisely the families

\[
\{g_k\}_{k=1}^\infty = \left\{ \hat{f}_k + h_k - \sum_{i=1}^\infty \hat{f}_i(x_i)h_i \right\}_{k=1}^\infty,
\]

where, \( \{h_k\}_{k=1}^\infty \) is an \( X_d \)-Bessel sequence for \( X \) and \( \{\hat{f}_i\}_{i=1}^\infty \) denotes the canonical dual Banach frame of \( \{x_i\}_{i=1}^\infty \) in \( X \) with respect to \( X_d \).

**Proof:** (i) By Theorem 2.6 and Theorem 2.7, we can characterize the dual Banach frames of \( \{f_i\}_{i=1}^\infty \) in \( X' \) with respect to \( X_d' \) as all families of the form

\[
\{z_k\}_{k=1}^\infty = \left\{ S + W(I_{X_d} - U_{X_d})e_i \right\}_{k=1}^\infty,
\]

where \( W : X_d \to X \) is a bounded operator, or equivalently an operator of the form \( W(c) = \sum_{i=1}^\infty c_i y_i \), where \( \{y_k\}_{k=1}^\infty \) is a \( X_d' \)-Bessel sequence for \( X' \). Inserting this expression for \( W \) we obtain

\[
\{z_k\}_{k=1}^\infty = \left\{ S(e_i) + W(e_i) - WUS(e_i) \right\}_{k=1}^\infty
\]

\[
= \left\{ \hat{x}_k + y_k - \sum_{i=1}^\infty f_i(\hat{x}_i)y_i \right\}_{k=1}^\infty.
\]

The proof for the statement (ii) is analogous.

**Definition 2.9:** A nonzero operator \( \Lambda \in B(X,Y) \) is called a left divisor of zero if there exists a nonzero operator \( \Gamma \in B(Y,X) \) such that \( \Lambda \Gamma = 0 \), similarly a nonzero operator \( \Lambda \in B(X,Y) \) is called a right divisor of zero if there exists a nonzero operator \( \Gamma \in B(Y,X) \) such that \( \Gamma \Lambda = 0 \).

**Lemma 2.10:** Let \( \{f_i\}_{i=1}^\infty \) and \( \{x_i\}_{i=1}^\infty \) be Banach frames for \( X, X' \) with respect to \( X_p, X_p' \), then the analysis and synthesis operators of them are right and left divisors of zero in \( B(X, X_p), B(X_p', X) \) respectively.

**Proof:** Suppose that \( U, V \) are the analysis and synthesis operators of \( \{f_i\}_{i=1}^\infty \) and \( \{x_i\}_{i=1}^\infty \) respectively.

Letting \( \Lambda : X_p' \to X \) and \( \Gamma : X \to X_p' \) by \( \Lambda = I_X - U \), and \( \Gamma = I_X - S \), where \( S, S \) denote the reconstruction operators of \( \{f_i\}_{i=1}^\infty \). Then we have \( \Lambda U = 0 \) and \( V \Gamma = 0 \).

This theorem is another characterization from the dual Banach frames by the family of left and right divisors of zero.

**Theorem 2.11:** Let \( \{f_i\}_{i=1}^\infty \) and \( \{x_i\}_{i=1}^\infty \) be Banach frames for \( X, X' \) with respect to \( X_p', X_p \) with the analysis and synthesis operators \( U, V \), respectively. Then the following holds:

(i) There exists an one to one correspondence between dual Banach frames of \( \{f_i\}_{i=1}^\infty \) in \( X' \) with respect to \( X_d' \) and the bounded operators \( \Lambda : X_d \to X \) such that \( \Lambda U = 0 \).

(ii) There exists an one to one correspondence between dual Banach frames of \( \{x_i\}_{i=1}^\infty \) in \( X \) with respect to \( X_d \) and the bounded operators \( \Gamma : X \to X_d' \) such that \( \Gamma V = 0 \).

**Proof:** (i) Let \( \{y_i\}_{i=1}^\infty \) be a dual Banach frame of \( \{f_i\}_{i=1}^\infty \) in \( X' \) with respect to \( X_d' \) with the synthesis operator \( W \). Define \( \Lambda : X_d \to X \) by \( \Lambda = W - S \), where \( S \) denotes the reconstruction operator of \( \{f_i\}_{i=1}^\infty \). Clearly, \( \Lambda \) is a bounded operator and by using Lemma 2.2 we have

\[
\Lambda U = WU - SU = 0
\]

For the opposite implication, suppose that \( \Lambda \) is a bounded operator from \( X_d \) in \( X \) such that \( \Lambda U = 0 \). Letting \( y_i = S \epsilon_i + \Lambda \epsilon_i \) for \( i \in I \) where \( \{\epsilon_i\}_{i=1}^\infty \) denotes the Schauder basis of the canonical unit vectors in \( X_d' \). As the proof of Proposition 2.4, \( \{y_i\}_{i=1}^\infty \) is a \( X_d' \)-Bessel sequence for \( X' \) and for every \( x \in X \) we have

\[
\sum_{i=1}^\infty f_i(x)y_i = \sum_{i=1}^\infty f_i(x)S \epsilon_i + \sum_{i=1}^\infty f_i(x)\Lambda \epsilon_i = SUx + \Lambda UX = x.
\]

This shows that \( \{y_i\}_{i=1}^\infty \) is a dual Banach frame of \( \{f_i\}_{i=1}^\infty \) in \( X' \) with respect to \( X_d' \).
(ii) The proof is similar to (i).

The following theorem is a perturbation result of dual Banach frames.

**Theorem 2.12:** Let \( \{f_i\}_{i\in I}, \{f'_i\}_{i\in I} \) be Banach frames for \( X \) with respect to \( X_d \) and let \( \{x_i\}_{i\in I} \) be a dual Banach frame of \( \{f_i\}_{i\in I} \) in \( X' \) with respect to \( X_d' \). If \( \{f_i - f'_i\}_{i\in I} \) and \( \{\tilde{x}_i - x'_i\}_{i\in I} \) are two \( X_d'Bessel \) \( X_d' \) Bessel sequences for \( X, X' \) with sufficiently small Bessel bounds \( \epsilon \). Then there exists a dual Banach frame \( \{x'_i\}_{i\in I} \) in \( X' \) with respect to \( X_d' \) such that \( \{x_i - x'_i\}_{i\in I} \) is a \( X_d'Bessel \) sequence in \( X' \) and its bound is a multiple of \( \epsilon \).

**Proof:** Suppose that \( U, S \) and \( U', S' \) are the analysis and reconstruction operators of \( \{f_i\}_{i\in I}, \{f'_i\}_{i\in I} \) then \( \tilde{x}_i = S_i e, \tilde{x}'_i = S'_i e \) for all \( i \in I \), where \( \{\tilde{e}_i\}_{i\in I} \) denotes the Schauder basis of the canonical unit vectors in \( X_d \). By the Theorem 2.11 there exists a bounded operator \( \Lambda : X \rightarrow X \) such that \( \Lambda U = 0 \) and \( x_i = \tilde{x}_i + \Lambda e_i, i \in I \). Defining \( y_i = \tilde{y}_i + \Lambda e_i \), it is easy to check that \( \{y_i\}_{i\in I} \) is a \( X_d'Bessel \) sequence in \( X' \). Denoting the synthesis operator of \( \{y_i\}_{i\in I} \) by \( V \), we claim that the bounded operator \( \Gamma x = VU'x = \sum_{i\in I} f'_i(x)y_i \) is invertible. In fact, for any \( x \in X \), we have

\[
||x - \Gamma x|| = ||x - \sum_{i\in I} f'_i(x)y_i|| = ||\Lambda U'x|| = ||\Lambda U'x - \Lambda U_x|| \leq ||\Lambda|| ||U'x - Ux|| \leq \epsilon ||\Lambda|| ||x||.
\]

Therefore, if \( \epsilon ||\Lambda|| < 1 \), then \( \Gamma \) is invertible and we obtain

\[
||\Gamma^{-1}|| \leq \frac{1}{1 - \epsilon ||\Lambda||} < \frac{1}{1 - \epsilon ||\Lambda||} \quad \text{and} \quad ||I_x - \Gamma^{-1}|| < \frac{\epsilon}{1 - \epsilon ||\Lambda||}.
\]

Put \( x'_i = \Gamma^{-1}y_i \) for all \( i \in I \) it is trivial that \( \{x'_i\}_{i\in I} \) is a \( X_d'Bessel \) sequence in \( X' \) and we see from \( \Gamma x = \sum_{i\in I} f'_i(x)y_i \) that \( x = \sum_{i\in I} f'_i(x)x'_i \). Hence \( \{x'_i\}_{i\in I} \) is a dual Banach frame for \( \{f'_i\}_{i\in I} \) in \( X' \) with respect to \( X_d' \).

On the other hand, for all \( c \in X_d \) we have:

\[
||\sum_{i\in I} c_i (x_i - x'_i)||_X = ||\sum_{i\in I} C_i(x_i - \Gamma^{-1}x_i + \Gamma^{-1}x_i - \Gamma^{-1}y_i)||_X \leq ||I_x - \Gamma^{-1}|| ||\sum_{i\in I} c_i (x_i - y'_i)||_X + ||\sum_{i\in I} C_i (x_i - y'_i)||_X.
\]

\[
\leq \frac{\epsilon}{1 - \epsilon ||\Lambda||} ||\sum_{i\in I} c_i (x_i - y'_i)||_X + \frac{1}{1 - \epsilon ||\Lambda||} ||\sum_{i\in I} c_i (x_i - y_i)||_X.
\]

This completes the proof.

**Corollary 2.13:** Let \( \{x_i\}_{i\in I}, \{x'_i\}_{i\in I} \) be Banach frames for \( X' \) with respect to \( X_d' \) and let \( \{f_i\}_{i\in I} \) be a dual Banach frame of \( \{x_i\}_{i\in I} \) in \( X \) with respect to \( X_d \). If \( \{f_i - f'_i\}_{i\in I} \) and \( \{x_i - x'_i\}_{i\in I} \) are two \( X_d'Bessel \) \( X_d' \) Bessel sequences for \( X, X' \) with sufficiently small Bessel bounds \( \epsilon \). Then there exists a dual Banach frame \( \{f'_i\}_{i\in I} \) for \( \{x'_i\}_{i\in I} \) in \( X \) with respect to \( X_d \) such that \( \{f_i - f'_i\}_{i\in I} \) is a \( X_d'Bessel \) sequence in \( X \) and its bound is a multiple of \( \epsilon \).

**Proof:** The proof is similar to Theorem 2.12.

## 3. Generalized Dual Banach Frames and Perturbation Results

In order to apply the dual frame expansions Hilbert space Christensen and Laugesen7, introduced the concepts of pseudo-duals and approximate duals for frames in Hilbert spaces. In this section we generalize this concepts in Banach spaces and examines their properties.

**Definition 3.1:** Suppose that \( \{f_i\}_{i\in I}, \{x_i\}_{i\in I} \) are \( X_d'Bessel \) and \( X_d'Bessel \) sequences for \( X, X' \) with analysis and synthesis operators \( U, V \) respectively. Then

(i) \( \{f_i\}_{i\in I} \) is a pseudo-dual Banach frame for \( \{x_i\}_{i\in I} \) in \( X \) with respect to \( X_d \) if \( UV \) is a bijection on \( X \).

(ii) We say that \( \{f_i\}_{i\in I} \) is an approximate dual Banach frame for \( \{x_i\}_{i\in I} \) in \( X \) with respect to \( X_d \) if \( \|I_x - UV\| < 1 \).

Note that if \( \{f_i\}_{i\in I} \) is a pseudo-dual Banach frame for \( \{x_i\}_{i\in I} \) in \( X \) with respect to \( X_d \), then

\[
x = \sum_{i\in I} f_i(x)(UV)^{-1}x, \quad \forall x \in X
\]

Thus \( \{f_i\}_{i\in I} \) is a dual Banach frame for \( \{(UV)^{-1}x_i\}_{i\in I} \) in \( X \) with respect to \( X_d \). From here, a standard argument shows that \( \{f_i\}_{i\in I} \) is a Banach frame for \( X \) with respect...
to $X'_d$. By symmetry $\{x'_i\}_{i=1}^n$ is also a Banach frame for $X'$ with respect to $X'_d$. Furthermore, if $\{f'_i\}_{i=1}^n$ is an approximate dual Banach frame for $\{x'_i\}_{i=1}^n$ in $X$ with respect to $X'_d$. Since the condition $\|I_x - VU\| \leq 1$ implies that the operator $VU$ is a bijection on $X$. Thins every approximate dual Banach frame is a pseudo-dual Banach frame.

The next lemma follows immediately from the definition. We leave the proof to interested readers.

**Lemma 3.2:** Let $\{f_i\}_{i=1}^n, \{x_i\}_{i=1}^n$ be $X'_d$-Bessel and $X'_d$-Bessel sequences for $X, X^*$ with analysis and synthesis operators $U, V$ respectively. Then the following statements are equivalent:

(i) $\{f_i\}_{i=1}^n$ is a pseudo-dual Banach frame for $\{x_i\}_{i=1}^n$ in $X$ with respect to $X'_d$.

(ii) $x = \sum_{i=1}^n f_i((VU')^{-1} x) f_i = \sum_{i=1}^n f_i(x)(VU')^{-1} x_i \quad \forall x \in X$.

(iii) $f = \sum_{i=1}^n f_i((VU')^{-1} x) x_i = \sum_{i=1}^n f_i(x)(VU')^{-1} f_i \quad \forall x \in X'$.

(iv) $f(x) = \sum_{i=1}^n f((VU')^{-1} x) f_i(x) = \sum_{i=1}^n f(x_i) f((VU')^{-1} x_i), \quad \forall x \in X, f \in X'$.

The next theorem shows that approximate dual Banach frames are stable under small perturbations of the Banach frame elements so that Theorem 2.2 obtained in is a special case of it.

**Theorem 3.3:** Let $\{f_i\}_{i=1}^n$ be a Banach frame for $X$ with respect to $X'_d$ with the analysis and reconstruction operators $U, S$. Assume that $\{g_i\}_{i=1}^n \subseteq X^*$ and there exist $\lambda, \mu \geq 0$ such that

(i) $2(\lambda \|U\| + \mu)\|S\| < 1$.

(ii) $\|f(x) - g(x)\|_{X_d} \leq \lambda \|f(x)\|_{X_d} + \mu \|x\|_{X_d}$, for all $x \in X$. Then $\{g_i\}_{i=1}^n$ is a Banach frame for $X$ with respect to $X'_d$ and $\{f_i\}_{i=1}^n, \{g_i\}_{i=1}^n$ are approximate dual Banach frames of $\{x_i\}_{i=1}^n, \{\tilde{y}_i\}_{i=1}^n$ in $X$ with respect to $X'_d$ respectively, where $\{x_i\}_{i=1}^n, \{\tilde{y}_i\}_{i=1}^n$ are the canonical dual Banach frame of $\{f_i\}_{i=1}^n, \{g_i\}_{i=1}^n$ in $X'$ with respect to $X'_d$.

**Proof:** Let $U'_d$ be the analysis operator of $\{g_i\}_{i=1}^n$, from the hypotheses we have

\[
\|U'_d x\|_{X'_d} \leq \|U'_d x - U'_d x\|_{X'_d} + \|U'_d x\|_{X'_d} 
\leq (\lambda + 1)\|U\| + \mu \|x\|_{X'},
\]

for all $x \in X$. This establishes the upper frame bound for $\{g_i\}_{i=1}^n$. On the other hand, from $SU = I_X$ we have $\|I_x - SU_U S_U|S\| \|U - U| < 1$ which follows that $\{g_i\}_{i=1}^n$ is an approximate dual Banach frame of $\{x_i\}_{i=1}^n$ and so

\[
\frac{1}{1 - \lambda \|U\| + \mu} \leq \frac{1}{1 - \lambda \|U\| + \mu} |S| < 1.
\]

**Corollary 3.4:** Let $\{f_i\}_{i=1}^n$ be a dual Banach frame of $\{x_i\}_{i=1}^n$ in $X$ with respect to $X'_d$ with the analysis and synthesis operators $U, V$. Assume that $\{g_i\}_{i=1}^n$ is a sequence $X^*$ in $X'$ and there exist $\lambda, \mu \geq 0$ such that

(i) $2(\lambda \|U\| + \mu)\|V\| < 1$.

(ii) $\|f(x) - g(x)\|_{X_d} \leq \lambda \|f(x)\|_{X_d} + \mu \|x\|_{X_d}$.

Then $\{g_i\}_{i=1}^n$ is an approximate dual Banach frame for $\{x_i\}_{i=1}^n$ in $X$ with respect to $X'_d$.

**Proof:** The proof is similar to Theorem 3.3.

**Theorem 3.5:** Let $\{f_i\}_{i=1}^n$ be a dual Banach frame for $\{x_i\}_{i=1}^n$ in $X$ with respect to $X'_d$ with the analysis and synthesis operators $U, V$. Assume that $\{y_i\}_{i=1}^n$ is a sequence in $X'$. Then $\{y_i\}_{i=1}^n$ is a Banach frame for $\{x_i\}_{i=1}^n$ in $X$ with respect to $X'_d$.

**Proof:** We obtain

\[
\|I_x - SU_U S_U|S\| \|U - U| < 1.
\]

This completes the proof.
Proof: The hypotheses given imply that the series $\sum_{i=1}^{\infty} f_i(x) y_i$ is convergent in $X$ for every $x \in X$. Thus the operator $\Lambda : X \to X$ defined by $\Lambda x = \sum_{i=1}^{\infty} f_i(x) y_i$ is bounded and holds
\[
\|x - \Lambda x\|_X \leq \lambda \|x\|_X + \mu \|U x\|_X \leq (\lambda + \mu \|U\|) \|x\|_X.
\]

From this we have $\|I - \Lambda\| < 1$, which follows that is an approximate dual Banach frame for $\{y_i\}_{i \in I}$ in $X$ with respect to $X_d$ and $\|\Lambda^{-1}\| \leq \frac{1}{1 - (\lambda + \mu \|U\|)}$. Therefore if we define $g_i = (\Lambda^{-1})^*(f_i), i \in I$, then $\{g_i\}_{i \in I}$ is a dual Banach frame for $\{y_i\}_{i \in I}$ in $X$ with respect to $X_d$. Let $U_1$ be the analysis operator of $\{g_i\}_{i \in I}$. Then we have
\[
\|I - VU_1\| = \|\sum_{i=1}^{\infty} f_i(x_i) y_i\| = \|\Lambda^{-1}\| \|x\|_X < 1.
\]

Therefore $\{g_i\}_{i \in I}$ is an approximate dual Banach frame for $\{x_i\}_{i \in I}$ in $X$ with respect to $X_d$.

Corollary 3.6: Let $\{f_i\}_{i \in I}$ be a dual Banach frame for $\{x_i\}_{i \in I}$ in $X$ with respect to $X_d$ and the analysis and synthesis operators $U, V$. Assume that $\{y_i\}_{i \in I}$ is a sequence in $X$ such that
\[
\left(\|x_i - y_i\|_X \in X_d \quad \text{and} \quad (1 + \|V\| \|U\|) \|x_i\|_X \right) \quad \|x_i - y_i\|_X < 1.
\]

Then $\{f_i\}_{i \in I}$ is an approximate dual Banach frame for $\{y_i\}_{i \in I}$ in $X$ with respect to $X_d$ and there exists a $X_d$-Bessel sequence $\{g_i\}_{i \in I}$ for $X$ such that $\{g_i\}_{i \in I}$ is an approximate dual Banach frame for $\{x_i\}_{i \in I}$ in $X$ with respect to $X_d$.

Proof: Since for every $c \in X_d$, we have
\[
\|\sum_{i=1}^{\infty} c_i(x_i - y_i)\|_X \leq \|\sum_{i=1}^{\infty} c_i\| X_d \|x_i - y_i\|_X \leq \|c\| X_d.
\]

Therefore the result follows from Theorem 3.5 with $\lambda = 0$ and $\mu = \|\sum_{i=1}^{\infty} c_i\| X_d$.

A $BK$-space $X_d$ is solid if whenever $\{b_i\}_{i \in I}, \{c_i\}_{i \in I}$ are sequences with $\|c_i\|_{X_d} \leq \|b_i\|_{X_d}$ and $\|b_i\|_{X_d} \leq \|c_i\|_{X_d}$, then it follows that $\|b_i\|_{X_d} \leq \|b_i\|_{X_d}$. Note that if the canonical unit vectors $e_i, i \in I$ form a Schauder basis for $X_d$ then $X_d$ is solid.

Corollary 3.7: Let $\{f_i\}_{i \in I}$ be a dual Banach frame for $\{x_i\}_{i \in I}$ in $X$ with respect to $X_d$ within the analysis and synthesis operators $U, V$. Assume that there exists a family $\{\alpha_i\}_{i \in I}$ of bounded operators on $X$ and scalars $\alpha_i$ such that $\alpha_i = \sup_{x \in X} \|\alpha_i x\| < \infty$ for all $x \in X$ and $\sum_{i=1}^{\infty} \alpha_i \|\lambda_j\| < 1$. Then $\{f_i\}_{i \in I}$ is an approximate dual Banach frame for $\{y_i\}_{i \in I}$ in $X$ with respect to $X_d$ and there exists a $X_d$-Bessel sequence $\{g_i\}_{i \in I}$ for $X$ such that $\{g_i\}_{i \in I}$ is an approximate dual Banach frame for $\{x_i\}_{i \in I}$ in $X$ with respect to $X_d$.

Proof: For all $j \in I$ and $c \in X_d$, we have
\[
\|\sum_{i=1}^{\infty} \alpha_i g_i(x_i)\|_X = \sup_{|I|} \|\sum_{i=1}^{\infty} \alpha_i f_i(x_i)\|_X \\ \leq \sup_{|I|} \|V(f)\|_X \|\sum_{i=1}^{\infty} \alpha_i g_i(x_i)\|_X \\ \leq \alpha_j \|\sum_{i=1}^{\infty} \alpha_i c_i\|_X.
\]

This yields
\[
\|\sum_{i=1}^{\infty} c_i(x_i - y_i)\|_X = \|\sum_{i=1}^{\infty} \alpha_i \lambda_j x_i\|_X \\ = \|\sum_{i=1}^{\infty} \alpha_i g_i(x_i)\|_X \\ \leq \|\sum_{i=1}^{\infty} \alpha_i \| \|\sum_{i=1}^{\infty} \alpha_i g_i(x_i)\|_X \\ \leq \|c\|_X \|\sum_{i=1}^{\infty} \alpha_i \| \|\lambda_j\|.
\]

Now the result follows from Theorem 3.5 with $\lambda = 0$ and $\mu = \|\sum_{i=1}^{\infty} \alpha_i \| \|\lambda_j\|$.

Proposition 3.8: Let $\{f_i\}_{i \in I}$ be an approximate dual Banach frame for $\{x_i\}_{i \in I}$ in $X$ with respect to $X_d$ with the analysis and synthesis operators $U, V$, respectively. Then the following holds:

(i) $\{(U^* V)^{-1} f_i\}_{i \in I}$ is a dual Banach frame for $\{x_i\}_{i \in I}$ in $X$ with respect to $X_d$ and
\[
(U^* V)^{-1} f_i = f_i + \sum_{n=1}^{\infty} (I - U^* V)^n f_i.
\]

(ii) For fixed $n \in N$, consider the partial sum $f_n = f_i + \sum_{n=1}^{\infty} (I - U^* V)^n f_i$. is an approximate dual Banach frame of $\{x_i\}_{i \in I}$ in $X$ with respect to $X_d$. Denoting its associated synthesis operator by $U_n$, we have
\[
\|I - U_n^* V\| \leq \|I - U^* V\| \to 0 \quad \text{as} \quad n \to \infty.
\]

Proof: (i) If $\{f_i\}_{i \in I}$ is an approximate dual Banach frame for $\{x_i\}_{i \in I}$, then the operator $VU$ is a bijection on $X$ and for all $x \in X$ we have
\[
\|f_i(x)\|_X \leq \|U(x)\|_X \quad \text{for} \quad i \in I.
\]

(ii) For fixed $n \in N$, consider the partial sum $f_n = f_i + \sum_{n=1}^{\infty} (I - U^* V)^n f_i$. is an approximate dual Banach frame of $\{x_i\}_{i \in I}$ in $X$ with respect to $X_d$. Denoting its associated synthesis operator by $U_n$, we have
\[
\|I - U_n^* V\| \leq \|I - U^* V\| \to 0 \quad \text{as} \quad n \to \infty.
Then defined by is a family of dual we have

\[ x = (VU)(VU)^{-1}x = \sum_{i \in I} f_i((VU)^{-1}x)x_i \]

Moreover, the inverse of \(UV\) can be written as follows:

\[ (VU)^{-1} = (I_X - (I_X - VU))^{-1} = I_X + \sum_{n=1}^\infty (I_X - VU)^n. \]

From this the result in (i) follows.

(ii) For any \( x \in X \) we have

\[ (I_X - VU_n)x = \]

\[ x - \sum_{i \in I} f_i(x)x_i = x - \sum_{i \in I} (I_X - U^*V^*)(f_i)(x)x_i \]

\[ = x - \sum_{j=0}^n VU(I_X - VU)^jx = x - \sum_{j=0}^n (I_X - (I_X - VU))(I_X - VU)^jx \]

\[ = (I_X - VU)^{n+1}x. \]

Thus,

\[ \|I_X - VU_n\| = \|(I_X - VU)^{n+1}\| \leq \|I_X - VU\| < 1. \]

**Proposition 3.9:** Let \( \{x_i\}_{i \in I}, \{y_i\}_{i \in I} \) be \( X_d \)-Riesz bases for \( X \) with the canonical dual Banach frames \( \{\tilde{f}_i\}_{i \in I}, \{\tilde{g}_i\}_{i \in I} \) respectively. Then \( \{\tilde{f}_i\}_{i \in I} \) is a pseudo-dual Banach frame for \( \{y_i\}_{i \in I} \) in \( X \) with respect to \( X_d \).

**Proof:** If \( V_1 \) and \( V_2 \) are the synthesis operators of \( \{x_i\}_{i \in I}, \{y_i\}_{i \in I} \). Then \( V_1, V_2 \) are invertible and \( V_1^{-1}, V_2^{-1} \) so are the analysis operators of \( \{\tilde{f}_i\}_{i \in I}, \{\tilde{g}_i\}_{i \in I} \) respectively. Thus \( V_1V_2^{-1} \) is a bijection on \( X \) and for every \( x \in X \) we have:

\[ x = \sum_{i \in I} \tilde{f}_i(x)(V_2V_2^{-1})y_i = \sum_{i \in I} \tilde{f}_i((V_2V_2^{-1})x)y_i \]

From this the claim follows immediately.

The following result will show that the image of a dual Banach frame under a bounded invertible operators is a pseudo-dual Banach frame.

**Theorem 3.10:** Let \( \{f_i\}_{i \in I}, \{x_i\}_{i \in I} \) be \( X_d \)-Bessel and \( X_d' \)-Bessel sequences for \( X, X \) respectively, and let \( \{\alpha_j\}_{j \in J}, \{\gamma_j\}_{j \in J} \) be a set of complex numbers such that \( \sum_j \alpha_j = 0 \). Then the following holds:

(i) if \( \{f_j\}_{j \in J} | f_j \in X', 1 \leq j \leq N \} \) is a family of dual Banach frames for \( \{x_i\}_{i \in I} \) in \( X \) with respect to \( X_d \) and \( \Lambda : X \to X \) is an invertible operator. Then the sequence \( \{g_j\}_{j \in J} \) in \( X \) defined by \( g_j = \sum_{j=1}^N \alpha_j \Lambda^*(f_j) \) is a pseudo-dual Banach frame for \( \{x_i\}_{i \in I} \) in \( X \) with respect to \( X_d' \).

(ii) if \( \{f_i\}_{i \in I} \) is a dual Banach frame for the family \( \{x_j\}_{j \in J}, x_j \in X, 1 \leq j \leq N \} \) in \( X \) with respect to \( X_d \) and \( \Lambda : X \to X \) is an invertible operator. Then \( \{f_i\}_{i \in I} \) is also a pseudo-dual Banach frame for the sequence \( \{\gamma_j\}_{j \in J} \) defined by \( \gamma_j = \sum_{j=1}^N \alpha_j \Lambda(x_j) \) in \( X \) with respect to \( X_d' \).

**Proof:** (i) Let \( U \) and \( V \) be the analysis and synthesis operators of \( \{f_j\}_{j \in J} \) and \( \{x_j\}_{j \in J} \) respectively. Since \( \{f_j\}_{j \in J} \) and \( \{x_j\}_{j \in J} \) are dual Banach frames for \( \{x_j\}_{j \in J} \) and \( \{f_j\}_{j \in J} \) respectively, this claim follows immediately from the fact that \( \|I_X - VU_n\| = \|(I_X - VU)^{n+1}\| \leq \|I_X - VU\| < 1 \).

(ii) The proof is similar to (i).

**Proposition 3.11:** Let \( \{f_i\}_{i \in I} \) and \( \{x_i\}_{i \in I} \) be \( X_d \)-Bessel and \( X_d' \)-Bessel sequences for \( X, X \) and let \( \Lambda, \Gamma \) be two invertible operators on \( X \). Then a sequence \( \{f_i\}_{i \in I} \) is a pseudo-dual Banach frame for \( \{x_i\}_{i \in I} \) in \( X \) with respect to \( X_d \) if and only if \( \{\Gamma^{-1} f_i\}_{i \in I} \) is a pseudo-dual Banach frame for \( \{\Lambda x_i\}_{i \in I} \) in \( X \) with respect to \( X_d \).

**Proof:** Suppose that \( U, U_1 \) and \( V, V_1 \) are the analysis and synthesis operators of \( \{f_i\}_{i \in I}, \{\Gamma^{-1} f_i\}_{i \in I} \) and \( \{x_i\}_{i \in I}, \{\Lambda x_i\}_{i \in I} \) respectively. This claim follows immediately from the fact that for each \( x \in X \) we have:

\[ V_1U_1(x) = \sum_{i \in I} \Gamma^{-1}(f_i)(x)\Lambda x_i = \Lambda(\sum_{i \in I} f_i(\Gamma x)x_i) = AVU_1T_x. \]

This finishes the proof.

**Proposition 3.12:** Let \( \{f_i\}_{i \in I}, \{x_i\}_{i \in I} \) be two \( X_d \)-Bessel and \( X_d' \)-Bessel sequences for \( X, X \) and let \( \Lambda, \Gamma \) be invertible operators on \( X \). If \( \{f_i\}_{i \in I} \) is a dual Banach frame for \( \{x_i\}_{i \in I} \) in \( X \) with respect to \( X_d \), then \( \{\Gamma^{-1} f_i\}_{i \in I} \) is a pseudo-dual Banach frame for \( \{\Lambda x_i\}_{i \in I} \) in \( X \) with respect to \( X_d' \).

**Proof:** The hypotheses imply that \( x = \sum_{i \in I} f_i(x)x_i \). Therefore:

\[ \sum_{i \in I} \Gamma^{-1}(f_i)(x)\Lambda x_i = \Lambda(\sum_{i \in I} f_i(\Gamma x)x_i) = \Lambda\Gamma x. \]

From this the result follows at once.
The following we give a method for constructing a family of pseudo-dual Banach frames from a given Banach frame.

**Proposition 3.13:** Let \( \{f_i\}_{i \in I} \) be a pseudo-dual Banach frame for \( \{x_i\}_{i \in I} \) in \( X \) with respect to \( X_d \) with the analysis and synthesis operators \( U, V \), respectively. If \( \alpha, \beta \) are two complex numbers such that \( \alpha + \beta = 1 \). Then the sequence \( \{g_i\}_{i \in I} \) defined by \( g_i = \alpha f_i + \beta (VU)^* (\tilde{f}_i) \), is a pseudo-dual Banach frame for \( \{x_i\}_{i \in I} \) in \( X \) with respect to \( X_d \), where \( \{\tilde{f}_i\}_{i \in I} \) is the canonical dual Banach frame of \( \{x_i\}_{i \in I} \).

**Proof:** For every \( x \in X \) we have

\[
\sum_{i \in I} g_i(x) x_i = \alpha \sum_{i \in I} f_i(x) x_i + \beta \sum_{i \in I} (VU)^* (\tilde{f}_i)(x) x_i
\]

\[
= \alpha VU x + \beta \sum_{i \in I} \tilde{f}_i(VU)x_i = (\alpha + \beta) VU x = VU x.
\]

The following we give a necessary condition for pseudo-duality of the sum of two pseudo-dual Banach frames from a given Banach frame.

**Proposition 3.14:** Let \( \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \) be two pseudo-dual Banach frames for \( \{x_i\}_{i \in I} \) in \( X \) with respect to \( X_d \) with the analysis and synthesis operators \( U_1, U_2 \) and \( V \), respectively. If

\[
\|(VU_1)^{-1}\| ||| V |||| || U_2 || < 1,
\]

Then the sequence \( \{f_i + g_i\}_{i \in I} \) is a pseudo-dual Banach frame for \( \{x_i\}_{i \in I} \) in \( X \) with respect to \( X_d \).

**Proof:** Since \( \|(VU_1)^{-1}\| ||| V |||| || U_2 || < 1 \) hence the operator \( I + (VU_1)^{-1} VU_2 \) is invertible which follows that \( V(U_1 + U_2) \) is invertible. Further

\[
\sum_{i \in I} (f_i + g_i)(x) x_i = \sum_{i \in I} f_i(x) x_i + \sum_{i \in I} g_i(x) x_i = V(U_1 + U_2)x
\]

Therefore \( \{f_i + g_i\}_{i \in I} \) is a pseudo-dual Banach frame for \( \{x_i\}_{i \in I} \).

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## 5. References